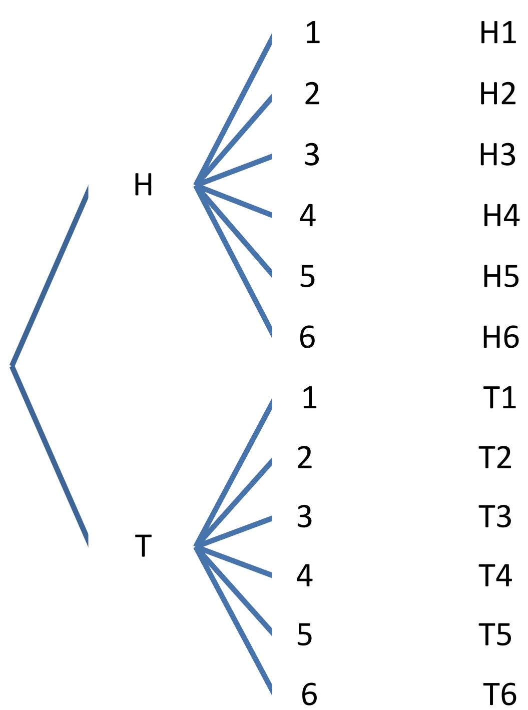
PRINCIPLES OF COUNTING

OBJECTIVES:

1. Construct a Tree Diagram
2. Apply the Multiplication Principle
3. Use Factorial Notation
4. Apply the Permutation Rule
5. Apply the Combination Rule

OBJECTIVE 1: CONSTRUCT A TREE DIAGRAM

A tree diagram is helpful in visualizing all possible outcomes, choices, or events. Suppose an experiment consists of tossing a coin, followed by rolling a die. To list all the possible outcomes, we can construct a tree diagram.



H = Head
T = Tail

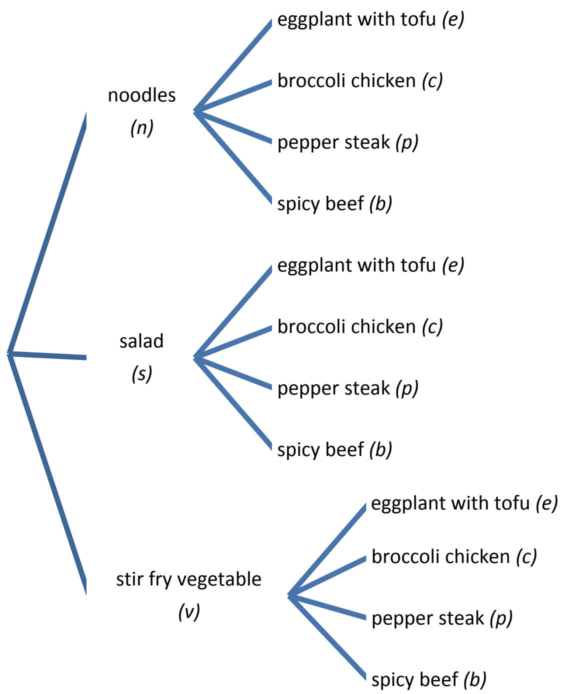
All possible outcomes are H1, H2, . . . , and T6. There are $2 \cdot 6 = 12$ possible outcomes.

EXAMPLE 1

A restaurant has a lunch special menu. The lunch special will be served on one plate; which includes rice and two items.
Item 1: Side: noodles, salad, or stir fry vegetable
Item 2: Entrée: eggplant with tofu, broccoli chicken, pepper steak, or spicy beef

Construct a tree diagram and find the total number of possible plate combinations.

Solution:



All possible plate combinations are *ne, nc, np, nb, se, sc, sp, . . ., vb*. There are $3 \cdot 4 = 12$ plate combinations for the lunch special.

OBJECTIVE 2: APPLY THE MULTIPLICATION PRINCIPLE:

Multiplication Principle

If a choice consists of *i* steps, of which the first step can be made in n_1 ways, the second in n_2 ways,..., and the last step in n_i ways, then the number of different ways possible choices can be made is

$$n_1 \cdot n_2 \cdot n_3 \cdots n_i$$

EXAMPLE 1

You have to choose a password containing exactly 4 characters, i.e. 2 letters (case sensitive), followed by 2 numbers, how many different choices of passwords are possible?

Solution:

Possible choices for the first letter (a – z, A – Z)		Possible choices for the first letter (a – z, A – Z)		Possible choices for the first number (0 – 9)		Possible choices for the second number (0 – 9)		
52	□	52	□	10	□	10	=	270,400

There are 270,400 choices of passwords.

EXAMPLE 2

Social security number consists of 9 digits. Assuming there are no restrictions on the number, how many different social security numbers can be issued?

Solution:

First digit	Second digit	Third digit	Fourth digit	. . .	Ninth digit	
(0 – 9)	(0 – 9)	(0 – 9)	(0 – 9)	. . .	(0 – 9)	
10 ·	10 ·	10 ·	10 ·	. . .	10	= 1,000,000,000

$$\underbrace{10 \cdot 10 \cdot 10 \cdot 10 \cdot \ldots \cdot 10}_{9 \text{ TIMES}} = 10^9$$

There are 1,000,000,000 possible social security numbers.

EXAMPLE 3

A short math quiz contains 2 true/false questions and 3 multiple choice questions consisting of a - d. In how many ways can a student answer all the questions?

Solution:

Question 1	Question 2	Question 3	Question 4	Question 5	
(T or F)	(T or F)	(a, b, c, or d)	(a, b, c, or d)	(a, b, c, or d)	
2 ·	2 ·	4 ·	4 ·	4	= 256

There are 256 different ways a student can answer the questions.

EXAMPLE 4

How many 3 digit numbers can be formed from numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 if no repetition is allowed?

Solution:
Since no repetition is allowed, each digit will have one fewer choice than the one directly preceding it.

First digit	Second digit	Third digit	
9 ·	8 ·	7	= 504

There are 504 different 3-digit numbers which can be formed from numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 if no repetition is allowed.

Objective 3: Use Factorial Notation

Factorial Notation
If n is a natural number, then n! (read as "n factorial"), is given by

$$n! = n(n-1)(n-2)(n-3)\cdots 2\cdot 1$$
$$0! = 1$$

Example 1
Calculate 5!

Solution:

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$$

Example 2
Calculate $\frac{15!}{12!}$

Solution:

$$\frac{15!}{12!} = \frac{15\cdot 14\cdot 13\cdot 12!}{12!} = 2730$$

Example 3
Find the number of ways we can arrange the letter A, B, C, D if no repetition is allowed.

Solution:
We can use either the multiplication principle or factorial notation.

Using Multiplication Principle

Since no repetition is allowed, we only have 3 choices for the second letter. We use one choice for the first letter so only 3 choices are left for the second letter, 2 choices left for the third and 1 choice left for the last letter.

First Letter (4 choices)		Second Letter (3 choices)		Third Letter (2 choices)		Fourth Letter (1 choice)		
4	$\cdot$	3	$\cdot$	2	$\cdot$	1	=	24

Using Factorial Notation

Factorial Notation yields the same result.

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Therefore, there are 24 ways to arrange the letters A, B, C, and D if no repetition is allowed.

Objective 4: Apply the Permutation Rule

Permutation Rule

The number of ways in which r objects can be selected in a specific order from n distinct objects when **order is important** and **no object is repeated** is given by the permutation ${}_nP_r$, where

$${}_nP_r = \frac{n!}{(n-r)!}$$

Note: ${}_nP_r$ is sometimes written as $P_{n,r}$ and is read as "n permute r" or "n select r".

Example 1

How many 3 digit numbers can be formed from numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 if no repetition is allowed?

Solution:

Understanding

Here we can use permutation with n = 9 and r = 3 because the order is important and there is no repetition. The number 215 is different from 125.

Using the Permutation Rule

$${}_9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!} = 504$$

Interpreting the result

There are 504 different 3-digit numbers which can be formed from numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 if no repetition is allowed.

Note: We can also use the multiplication principle to answer this question.

Example 2

A board of directors of an HOA (Home Owner's Association) consists of 8 people. How many ways can the HOA members pick a president, a secretary, and a treasurer from the board of directors?

Solution:

Understanding

We can use permutation with n = 8 and r = 3 because the order is important and there is no repetition.

The outcome that Andrew is chosen as the president, Billy as the secretary, and Cynthia as the treasurer is a different outcome from Billy as the president, Cynthia as the secretary, and Andrew as the treasurer; hence the order is important.

Each person can only hold one position. Someone who is chosen as the president cannot be chosen as a secretary. Therefore, there is no repetition in this case.

Using the Permutation Rule

$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{5!} = 336$$

Interpreting the Result
There are 336 different ways to choose a president, a secretary, and a treasurer from 8 members of the board of directors.

Note: We can also use the multiplication principle to answer this question.

OBJECTIVE 5: APPLY THE COMBINATION RULE

Combination rule

The number of ways in which *r* objects can be chosen from *n* different objects, when **order is <u>not</u> important** and **no object is repeated** is given by the **combination** $_nC_r$, where $_nC_r$ is

$$_nC_r = \frac{n!}{r!(n-r)!}$$

Note: $_nC_r$ is sometimes written as $\begin{pmatrix} n \\ r \end{pmatrix}$ or $C_{n,r}$ and is read as "*n* choose *r*".

EXAMPLE 1
How many ways can we choose 3 different color combinations from the primary colors of the rainbow, i.e. red, orange, yellow, green, blue, indigo, and violet?

Solution:

Understanding

We use combination with *n* = 7 and *r* =3 because the order is not important and there is no repetition. The color combination rbg is considered the same as rgb.

Using the Combination Rule

$$_7C_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2} = 35$$

Interpreting the Result
There are 35 different color combinations containing three colors out of the seven primary colors of rainbow.

EXAMPLE 2
A committee consisting 12 members must form a subcommittee consisting of 4 members. How many different subcommittees are possible?

Understanding
Here, we can use combination with n = 12 and r = 4. The order is not important because there is no ranking of subcommittee members. Also there is no repetition.

Using the Combination Rule

$$_{12}C_4 = \frac{12!}{4!(12-4)!} = \frac{12!}{4!8!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8!}{4\cdot 3\cdot 2\cdot 8!} = 495$$

Interpreting the Result
There are 495 possible subcommittees of size 4 chosen from the 12 members.

EXERCISES:

Constructing a Tree Diagram

In exercises 1 - 2, construct a tree diagram to represent all possible choices (or outcomes) and determine the total number of available choices (or outcomes).

1. A student goes to a coffee shop and has two choices for ordering coffee.
   - Choice 1: Size: small, medium, or large
   - Choice 2: Type of coffee drink: cappuccino, mocha, espresso, Americano, or Frappuccino
2. A health survey consists of three questions.
   - Question 1: Gender: male, female, or other
   - Question 2: Age group: under 18, 18 – 30, 31 – 50, above 50
   - Question 3: Has the person smoked: yes or no

Applying the Multiplication Principle

In exercises 3 - 7, apply the multiplication principle to answer the questions.

3. Tom's Pizza offers three sizes of pizza: small, medium, and large. There are 2 types of crusts and 12 types of toppings. How many ways can a person order a pizza with one topping?
4. How many different ways can we arrange the letters of the word PLANTS?
5. A tourist in Spain wants to visit 8 cities in 3 weeks. How many different routes are possible?
6. A luggage lock has 4 dials. Each dial has the digits 0 to 9. How many different combinations are there for the lock?
7. License plates in California consist of one number followed by three letters and three numbers. How many different license plates can be made?

Using the Factorial Notation

In Exercises 8 - 13, calculate the given factorial.

8. 5!
9. 0!
10. 9!
11. 11!
12. 1!
13. 15!

Using the Permutation Rule

In Exercises 14 - 27, calculate the given permutation. Express large values using E-notation with the mantissa rounded to two decimals.

14. ${}_{7}P_{2}$
15. ${}_{16}P_{4}$
16. ${}_{24}P_{4}$
17. ${}_{35}P_{10}$
18. ${}_{6}P_{6}$
19. ${}_{50}P_{49}$
20. ${}_{17}P_{15}$
21. ${}_{27}P_{7}$
22. ${}_{8}P_{7}$

23. A chess club has 24 members. How many different slates of candidates are possible if they must select a chairperson, a secretary, and a treasurer from the club members?
24. How many different ways can we arrange the letters of the word PLANTS?
25. A tourist in Spain wants to visit 8 cities in 3 weeks. How many different routes are possible?
26. There are 40 students in a class. The desks in the classroom are arranged so that each row consists of 8 desks. How many different seating arrangements of the first row are possible?
27. Twelve students participate in the local science contest. In how many ways can the students place first, second, and third?

Using the Combination Rule

In Exercises 28 - 42, calculate the given combination. Express large values using E-notation with the mantissa rounded to two decimals.

28. ${}_{5}C_{5}$
29. ${}_{16}C_{15}$
30. ${}_{24}C_{3}$
31. ${}_{15}C_{10}$
32. ${}_{26}C_{23}$
33. ${}_{40}C_{39}$
34. ${}_{17}C_{1}$
35. ${}_{27}C_{0}$
36. ${}_{8}C_{4}$

37. A committee consisting 22 members must form a subcommittee consisting of 4 members. How many different subcommittees are possible?
38. How many ways can an IRS agent select 5 tax returns out of 12 for an audit?
39. Seven candidates are running to become the council members of the city of Westlake Village. There are 3 open positions. How many different compositions of council members are there?
40. Arnold is going on a camping trip. He has 10 favorite shirts, but he plans to bring only 4 shirts. How many different combinations of shirts are possible?
41. Cynthia is creating a flower arrangement and wants to use 5 different types of flowers in her arrangement. If there are 10 types of flowers in the florist shop, how many ways can Cynthia select the flowers?
42. There are 40 students in a class. The desks in the classroom are arranged so that each row consists of 8 desks. How many different seating arrangements of the first row are possible?

Using the Multiplication Principles and Combination/Permutation Rule

In exercises 43 - 48, combine the multiplication principle and combination or permutation rule.

43. Tom's Pizza offers three sizes of pizza: small, medium, and large. There are 2 types of crusts and 12 types of toppings. How many ways can a person order a pizza with three toppings?
44. How many different ways can we arrange the letters of the word MISSISSIPPI?

45. To play Powerball, a person needs to select five numbers from 1 to 69 for the white balls; then select one number from 1 to 26 for the red Powerball. How many different ways can the Powerball numbers be picked?
46. The CEO of a company wants to visit the company's branch offices in New Mexico, Texas, and Louisiana. The company has 10 offices in New Mexico, 12 in Texas, and 5 in Louisiana. If the CEO wants to pick 5 in both New Mexico and Texas, and 3 in Louisiana to visit, how many different ways can he pick the branch offices to visit?
47. Arnold is going on a camping trip. He has 10 favorite shirts, but he plans to bring only 4 shirts. He has 5 different pairs of pants, but he wants to bring 2 pairs. How many different outfits can Arnold wear in his camping trip?
48. A jar of Halloween treats contains of 24 pieces of Hershey's chocolates and 12 pieces of Reese's peanut butter cups. In how many ways can Anna pull out 3 pieces of chocolates and 2 pieces of peanut butter cups?