SUPPLEMENT 2A

INTRODUCTION TO FUNCTIONS

This odd-looking headgear provides the user with a virtual world. (credit: fill/Pixabay)

Introduction

Imagine visiting a faraway city or even outer space from the comfort of your living room. It could be possible using virtual reality. This technology creates realistic images that make you feel as if you are truly immersed in the scene and even enable you to interact with them. It is being developed for fun applications, such as video games, but also for architects to plan buildings, car companies to design prototypes, the military to train, and medical students to learn.

Developing virtual reality devic es requires modeling the environment using graphs and mathematic al relationships. In this chapter, you will graph different relationships and learn ways to describe and analyze graphs. You will learn to identify the domain and range of a function.You will also learn how to perform function composition and how to find the inverse of a one-to-one function.

3.5 Relations and Functions

Learning Objectives

By the end of this section, you will be able to:

- Find the domain and range of a relation
- Determine if a relation is a function
- Find the value of a function

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $3x - 5$ when $x = -2$.
2. Evaluate $2x^2 - x - 3$ when $x = a$.
3. Simplify: $7x - 1 - 4x + 5$.

Find the Domain and Range of a Relation

As we go about our daily lives, we have many data items or quantities that are paired to our names. Our social security number, student ID number, email address, phone number and our birthday are matched to our name. There is a relationship between our name and each of those items.

When your professor gets her class roster, the names of all the students in the class are listed in one column and then the student ID number is likely to be in the next column. If we think of the correspondence as a set of ordered pairs, where the first element is a student name and the second element is that student's ID number, we call this a **relation**.

$$(\text{Student name, Student ID \#})$$

The set of all the names of the students in the class is called the **domain** of the relation and the set of all student ID numbers paired with these students is the range of the relation.

There are many similar situations where one variable is paired or matched with another. The set of ordered pairs that records this matching is a relation.

Relation

A **relation** is any set of ordered pairs, (x, y). All the *x*-values in the ordered pairs together make up the **domain**. All the *y*-values in the ordered pairs together make up the **range**.

EXAMPLE 3.42

For the relation $\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

Solution

$$\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$$

ⓐ The domain is the set of all *x*-values of the relation. $\{1, 2, 3, 4, 5\}$

ⓑ The range is the set of all *y*-values of the relation. $\{1, 4, 9, 16, 25\}$

TRY IT : : 3.83 For the relation $\{(1, 1), (2, 8), (3, 27), (4, 64), (5, 125)\}$:

ⓐ Find the domain of the relation.

ⓑ Find the range of the relation.

TRY IT : : 3.84 For the relation $\{(1, 3), (2, 6), (3, 9), (4, 12), (5, 15)\}$:

ⓐ Find the domain of the relation.

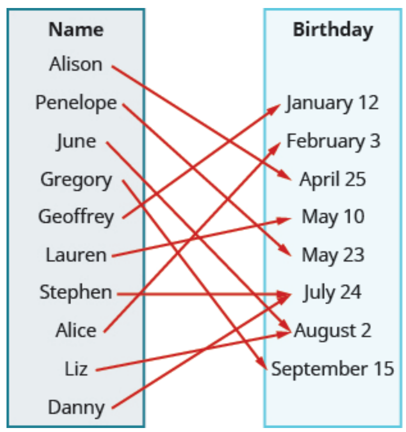
ⓑ Find the range of the relation.

Mapping

A **mapping** is sometimes used to show a relation. The arrows show the pairing of the elements of the domain with the elements of the range.

EXAMPLE 3.43

Use the **mapping** of the relation shown to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the range of the relation.



Solution

ⓐ The arrow shows the matching of the person to their birthday. We create ordered pairs with the person's name as the *x*-value and their birthday as the *y*-value.

{(Alison, April 25), (Penelope, May 23), (June, August 2), (Gregory, September 15), (Geoffrey, January 12), (Lauren, May 10), (Stephen, July 24), (Alice, February 3), (Liz, August 2), (Danny, July 24)}

ⓑ The domain is the set of all *x*-values of the relation.

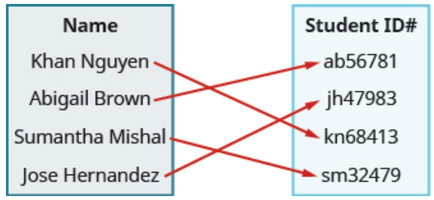
{Alison, Penelope, June, Gregory, Geoffrey, Lauren, Stephen, Alice, Liz, Danny}

ⓒ The range is the set of all *y*-values of the relation.

{January 12, February 3, April 25, May 10, May 23, July 24, August 2, September 15}

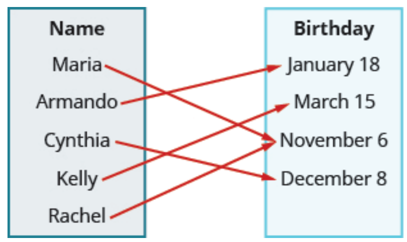
TRY IT : : 3.85

Use the mapping of the relation shown to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



TRY IT : : 3.86

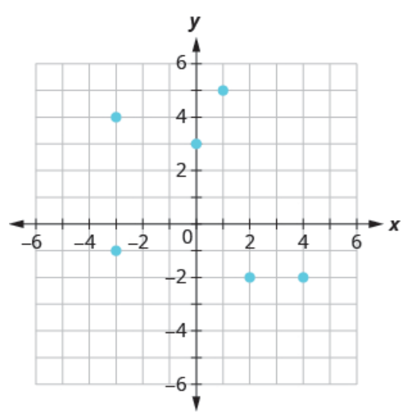
Use the mapping of the relation shown to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



A graph is yet another way that a relation can be represented. The set of ordered pairs of all the points plotted is the relation. The set of all *x*-coordinates is the domain of the relation and the set of all *y*-coordinates is the range. Generally we write the numbers in ascending order for both the domain and range.

EXAMPLE 3.44

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution

ⓐ The ordered pairs of the relation are: $\{(1, 5), (-3, -1), (4, -2), (0, 3), (2, -2), (-3, 4)\}$.

ⓑ The domain is the set of all *x*-values of the relation: $\{-3, 0, 1, 2, 4\}$.

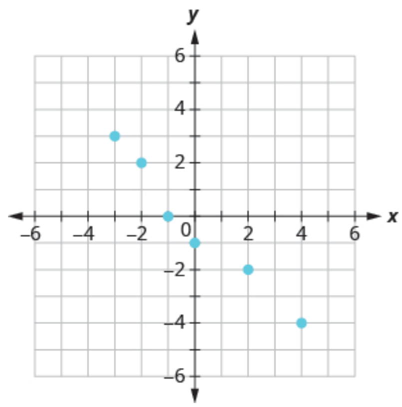
Notice that while -3 repeats, it is only listed once.

ⓒ The range is the set of all *y*-values of the relation: $\{-2, -1, 3, 4, 5\}$.

Notice that while -2 repeats, it is only listed once.

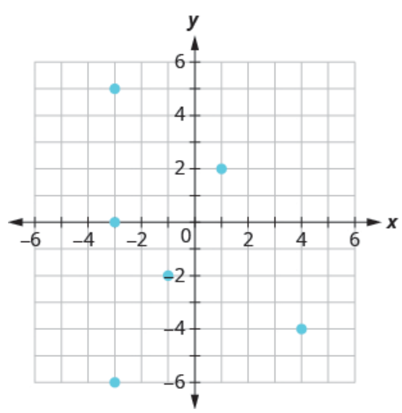
TRY IT : : 3.87

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



TRY IT : : 3.88

Use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.



Determine if a Relation is a Function

A special type of relation, called a **function**, occurs extensively in mathematics. A function is a relation that assigns to each element in its domain exactly one element in the range. For each ordered pair in the relation, each *x*-value is matched with only one *y*-value.

Function

A **function** is a relation that assigns to each element in its domain exactly one element in the range.

The birthday example from Example 3.43 helps us understand this definition. Every person has a birthday but no one has two birthdays. It is okay for two people to share a birthday. It is okay that Danny and Stephen share July 24th as their birthday and that June and Liz share August 2nd. Since each person has exactly one birthday, the relation in Example 3.43 is a function.

The relation shown by the graph in Example 3.44 includes the ordered pairs $(-3, -1)$ and $(-3, 4)$. Is that okay in a function? No, as this is like one person having two different birthdays.

EXAMPLE 3.45

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(-3, 27), (-2, 8), (-1, 1), (0, 0), (1, 1), (2, 8), (3, 27)\}$

ⓑ $\{(9,\ -3),\ (4,\ -2),\ (1,\ -1),\ (0,\ 0),\ (1,\ 1),\ (4,\ 2),\ (9,\ 3)\}$

Solution

ⓐ $\{(-3,\ 27),\ (-2,\ 8),\ (-1,\ 1),\ (0,\ 0),\ (1,\ 1),\ (2,\ 8),\ (3,\ 27)\}$

(i) Each *x*-value is matched with only one *y*-value. So this relation is a function.

(ii) The domain is the set of all *x*-values in the relation.
The domain is: $\{-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}$.

(iii) The range is the set of all *y*-values in the relation. Notice we do not list range values twice.
The range is: $\{27,\ 8,\ 1,\ 0\}$.

ⓑ $\{(9,\ -3),\ (4,\ -2),\ (1,\ -1),\ (0,\ 0),\ (1,\ 1),\ (4,\ 2),\ (9,\ 3)\}$

(i) The *x*-value 9 is matched with two *y*-values, both 3 and -3. So this relation is not a function.

(ii) The domain is the set of all *x*-values in the relation. Notice we do not list domain values twice.
The domain is: $\{0,\ 1,\ 2,\ 4,\ 9\}$.

(iii) The range is the set of all *y*-values in the relation.
The range is: $\{-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}$.

TRY IT : : 3.89

Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the function.

ⓐ $\{(-3,\ -6),\ (-2,\ -4),\ (-1,\ -2),\ (0,\ 0),\ (1,\ 2),\ (2,\ 4),\ (3,\ 6)\}$

ⓑ $\{(8,\ -4),\ (4,\ -2),\ (2,\ -1),\ (0,\ 0),\ (2,\ 1),\ (4,\ 2),\ (8,\ 4)\}$

TRY IT : : 3.90

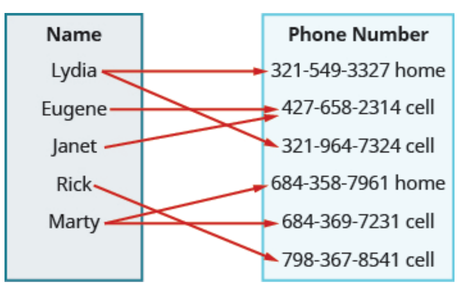
Use the set of ordered pairs to (i) determine whether the relation is a function (ii) find the domain of the relation (iii) find the range of the relation.

ⓐ $\{(27,\ -3),\ (8,\ -2),\ (1,\ -1),\ (0,\ 0),\ (1,\ 1),\ (8,\ 2),\ (27,\ 3)\}$

ⓑ $\{(7,\ -3),\ (-5,\ -4),\ (8,\ -0),\ (0,\ 0),\ (-6,\ 4),\ (-2,\ 2),\ (-1,\ 3)\}$

EXAMPLE 3.46

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



Solution

ⓐ Both Lydia and Marty have two phone numbers. So each *x*-value is not matched with only one *y*-value. So this relation is not a function.

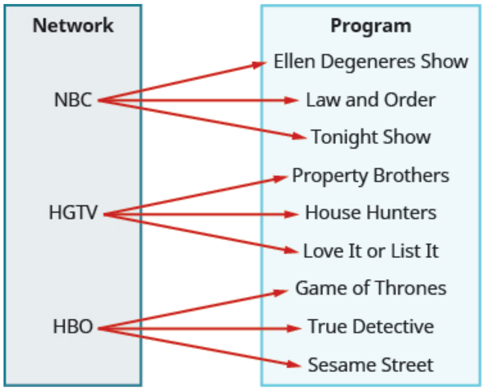
ⓑ The domain is the set of all *x*-values in the relation. The domain is: {Lydia, Eugene, Janet, Rick, Marty}

ⓒ The range is the set of all *y*-values in the relation. The range is:

$\{321\text{-}549\text{-}3327,\quad 427\text{-}658\text{-}2314,\quad 321\text{-}964\text{-}7324,\quad 684\text{-}358\text{-}7961,\quad 684\text{-}369\text{-}7231,\quad 798\text{-}367\text{-}8541\}$

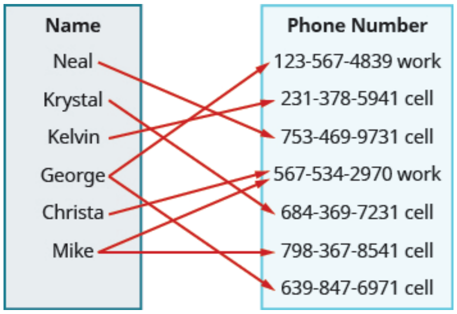
TRY IT : : 3.91

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



TRY IT : : 3.92

Use the mapping to ⓐ determine whether the relation is a function ⓑ find the domain of the relation ⓒ find the range of the relation.



In algebra, more often than not, functions will be represented by an equation. It is easiest to see if the equation is a function when it is solved for *y*. If each value of *x* results in only one value of *y*, then the equation defines a function.

EXAMPLE 3.47

Determine whether each equation is a function.

ⓐ $2x + y = 7$ ⓑ $y = x^2 + 1$ ⓒ $x + y^2 = 3$

Solution

ⓐ $2x + y = 7$

For each value of *x*, we multiply it by -2 and then add 7 to get the *y*-value

	$y = -2x + 7$
For example, if $x = 3$:	$y = -2 \cdot 3 + 7$
	$y = 1$

We have that when $x = 3$, then $y = 1$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓑ $y = x^2 + 1$

For each value of *x*, we square it and then add 1 to get the *y*-value.

	$y = x^2 + 1$
For example, if $x = 2$:	$y = 2^2 + 1$
	$y = 5$

We have that when $x = 2$, then $y = 5$. It would work similarly for any value of *x*. Since each value of *x*, corresponds to only one value of *y* the equation defines a function.

ⓒ

	$x + y^2 = 3$
Isolate the *y* term.	$y^2 = -x + 3$
Let's substitute $x = 2$.	$y^2 = -2 + 3$
	$y^2 = 1$
This give us two values for *y*.	$y = 1$ $y = -1$

We have shown that when $x = 2$, then $y = 1$ and $y = -1$. It would work similarly for any value of *x*. Since each value of *x* does not corresponds to only one value of *y* the equation does not define a function.

TRY IT : : 3.93 Determine whether each equation is a function.

ⓐ $4x + y = -3$ ⓑ $x + y^2 = 1$ ⓒ $y - x^2 = 2$

TRY IT : : 3.94 Determine whether each equation is a function.

ⓐ $x + y^2 = 4$ ⓑ $y = x^2 - 7$ ⓒ $y = 5x - 4$

Find the Value of a Function

It is very convenient to name a function and most often we name it *f*, *g*, *h*, *F*, *G*, or *H*. In any function, for each *x*-value from the domain we get a corresponding *y*-value in the range. For the function *f*, we write this range value *y* as $f(x)$. This is called function notation and is read *f* of *x* or the value of *f* at *x*. In this case the parentheses does not indicate multiplication.

Function Notation

For the function $y = f(x)$

f is the name of the function
x is the domain value
$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as *f* of *x* or the value of *f* at *x*.

We call *x* the independent variable as it can be any value in the domain. We call *y* the dependent variable as its value depends on *x*.

Independent and Dependent Variables

For the function $y = f(x)$,

x is the independent variable as it can be any value in the domain
y the dependent variable as its value depends on x

Much as when you first encountered the variable *x*, function notation may be rather unsettling. It seems strange because it is new. You will feel more comfortable with the notation as you use it.

Let's look at the equation $y = 4x - 5.$ To find the value of *y* when $x = 2,$ we know to substitute $x = 2$ into the equation and then simplify.

	$y = 4x - 5$
Let $x = 2.$	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5.$ To find the value when $x = 2,$ we write:

	$f(x) = 4x - 5$
Let $x = 2.$	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of *x* is called *evaluating the function.*

EXAMPLE 3.48

For the function $f(x) = 2x^2 + 3x - 1,$ evaluate the function.

ⓐ $f(3)$ ⓑ $f(-2)$ ⓒ $f(a)$

Solution

ⓐ

	$f(x) = 2x^2 + 3x - 1$
To evaluate $f(3),$ substitute 3 for *x*.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$
	$f(3) = 26$

ⓑ

	$f(x)=2x^2+3x-1$
To evaluate $f(-2)$, substitute -2 for x.	$f(-2)=2(-2)^2+3(-2)-1$
Simplify.	$f(-2)=2\cdot4+(-6)-1$
	$f(-2)=8+(-6)-1$
	$f(-2)=1$

ⓒ

	$f(x)=2x^2+3x-1$
To evaluate $f(a)$, substitute a for x.	$f(a)=2(a)^2+3\cdot a-1$
Simplify.	$f(a)=2a^2+3a-1$

TRY IT : : 3.95 For the function $f(x)=3x^2-2x+1$, evaluate the function.

ⓐ $f(3)$ ⓑ $f(-1)$ ⓒ $f(t)$

TRY IT : : 3.96 For the function $f(x)=2x^2+4x-3$, evaluate the function.

ⓐ $f(2)$ ⓑ $f(-3)$ ⓒ $f(h)$

In the last example, we found $f(x)$ for a constant value of x. In the next example, we are asked to find $g(x)$ with values of x that are variables. We still follow the same procedure and substitute the variables in for the x.

EXAMPLE 3.49

For the function $g(x)=3x-5$, evaluate the function.

ⓐ $g(h^2)$ ⓑ $g(x+2)$ ⓒ $g(x)+g(2)$

Solution

ⓐ

	$g(x)=3x-5$
To evaluate $g(h^2)$, substitute h^2 for x.	$g(h^2)=3h^2-5$
	$g(h^2)=3h^2-5$

ⓑ

	$g(x) = 3x - 5$
To evaluate $g(x+2)$, substitute $x+2$ for x.	$g(x+2) = 3(x+2) - 5$
Simplify.	$g(x+2) = 3x + 6 - 5$
	$g(x+2) = 3x + 1$

ⓒ

	$g(x) = 3x - 5$
To evaluate $g(x) + g(2)$, first find $g(2)$.	$g(2) = 3 \cdot 2 - 5$
	$g(2) = 1$
Now find $g(x) + g(2)$	$g(x) + g(2) = \underbrace{3x - 5}_{g(x)} + \underbrace{1}_{g(2)}$
Simplify.	$g(x) + g(2) = 3x - 5 + 1$
	$g(x) + g(2) = 3x - 4$

Notice the difference between part ⓑ and ⓒ. We get $g(x+2) = 3x + 1$ and $g(x) + g(2) = 3x - 4$. So we see that $g(x+2) \neq g(x) + g(2)$.

TRY IT : : 3.97 For the function $g(x) = 4x - 7$, evaluate the function.

ⓐ $g(m^2)$ ⓑ $g(x-3)$ ⓒ $g(x) - g(3)$

TRY IT : : 3.98 For the function $h(x) = 2x + 1$, evaluate the function.

ⓐ $h(k^2)$ ⓑ $h(x+1)$ ⓒ $h(x) + h(1)$

Many everyday situations can be modeled using functions.

EXAMPLE 3.50

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(5)$. Explain what this result means.

Solution

ⓐ The number of unread emails is a function of the number of days. The number of unread emails, *N*, depends on the number of days, *t*. Therefore, the variable *N*, is the dependent variable and the variable t is the independent variable.

ⓑ Find $N(5)$. Explain what this result means.

	$N(t) = 75 + 10t$
Substitute in $t = 5$.	$N(5) = 75 + 10 \cdot 5$
Simplify.	$N(5) = 75 + 50$
	$N(5) = 125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

TRY IT : : 3.99

The number of unread emails in Bryan's account is 100. This number grows by 15 unread emails a day. The function $N(t) = 100 + 15t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(7)$. Explain what this result means.

TRY IT : : 3.100

The number of unread emails in Anthony's account is 110. This number grows by 25 unread emails a day. The function $N(t) = 110 + 25t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

ⓐ Determine the independent and dependent variable.

ⓑ Find $N(14)$. Explain what this result means.

MEDIA : :

Access this online resource for additional instruction and practice with relations and functions.

- **Introduction to Functions (https://openstax.org/l/37introfunction)**

3.5 EXERCISES

Practice Makes Perfect

Find the Domain and Range of a Relation

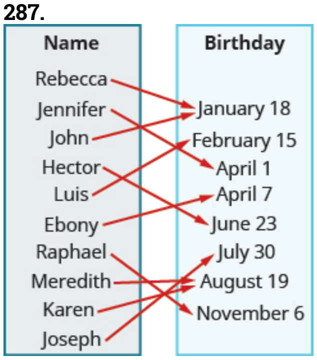
In the following exercises, for each relation ⓐ find the domain of the relation ⓑ find the range of the relation.

283. $\{(1,\ 4),\ (2,\ 8),\ (3,\ 12),\ (4,\ 16),\ (5,\ 20)\}$

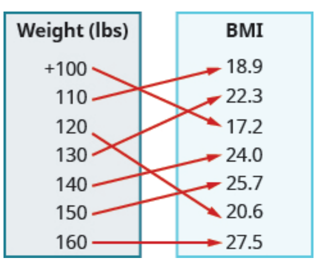
285. $\{(1,\ 7),\ (5,\ 3),\ (7,\ 9),\ (-2,\ -3),\ (-2,\ 8)\}$

In the following exercises, use the mapping of the relation to ⓐ list the ordered pairs of the relation, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

287.

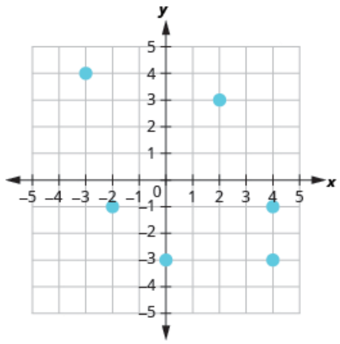


289. For a woman of height $5'4''$ the mapping below shows the corresponding Body Mass Index (BMI). The body mass index is a measurement of body fat based on height and weight. A BMI of $18.5-24.9$ is considered healthy.

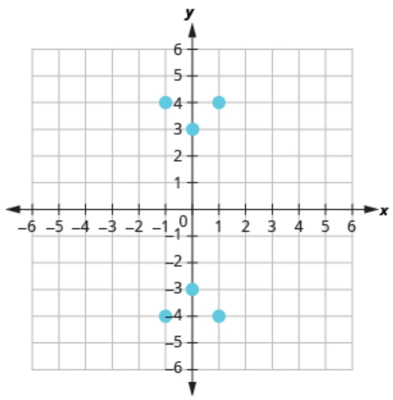


In the following exercises, use the graph of the relation to ⓐ list the ordered pairs of the relation ⓑ find the domain of the relation ⓒ find the range of the relation.

291.



293.



Determine if a Relation is a Function

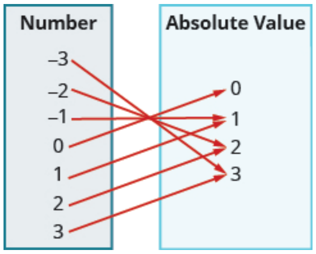
In the following exercises, use the set of ordered pairs to ⓐ determine whether the relation is a function, ⓑ find the domain of the relation, and ⓒ find the range of the relation.

295. $\{(-3,\ 9),\ (-2,\ 4),\ (-1,\ 1),\ (0,\ 0),\ (1,\ 1),\ (2,\ 4),\ (3,\ 9)\}$

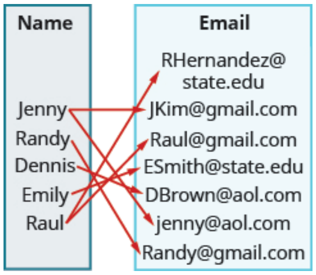
297. $\{(-3,\ 27),\ (-2,\ 8),\ (-1,\ 1),\ (0,\ 0),\ (1,\ 1),\ (2,\ 8),\ (3,\ 27)\}$

In the following exercises, use the mapping to ⓐ determine whether the relation is a function, ⓑ find the domain of the function, and ⓒ find the range of the function.

299.



301.



In the following exercises, determine whether each equation is a function.

303.

ⓐ $2x + y = -3$

ⓑ $y = x^2$

ⓒ $x + y^2 = -5$

305.

ⓐ $y - 3x^3 = 2$

ⓑ $x + y^2 = 3$

ⓒ $3x - 2y = 6$

Find the Value of a Function

In the following exercises, evaluate the function: ⓐ $f(2)$ ⓑ $f(-1)$ ⓒ $f(a)$.

307. $f(x) = 5x - 3$

309. $f(x) = -4x + 2$

311. $f(x) = x^2 - x + 3$

313. $f(x) = 2x^2 - x + 3$

In the following exercises, evaluate the function: ⓐ $g(h^2)$ ⓑ $g(x + 2)$ ⓒ $g(x) + g(2)$.

315. $g(x) = 2x + 1$

317. $g(x) = -3x - 2$

319. $g(x) = 3 - x$

3.6 Graphs of Functions

Learning Objectives

By the end of this section, you will be able to:

- Use the vertical line test
- Identify graphs of basic functions
- Read information from a graph of a function

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate: ⓐ 2^3 ⓑ 3^2.
2. Evaluate: ⓐ $|7|$ ⓑ $|-3|$.
3. Evaluate: ⓐ $\sqrt{4}$ ⓑ $\sqrt{16}$.

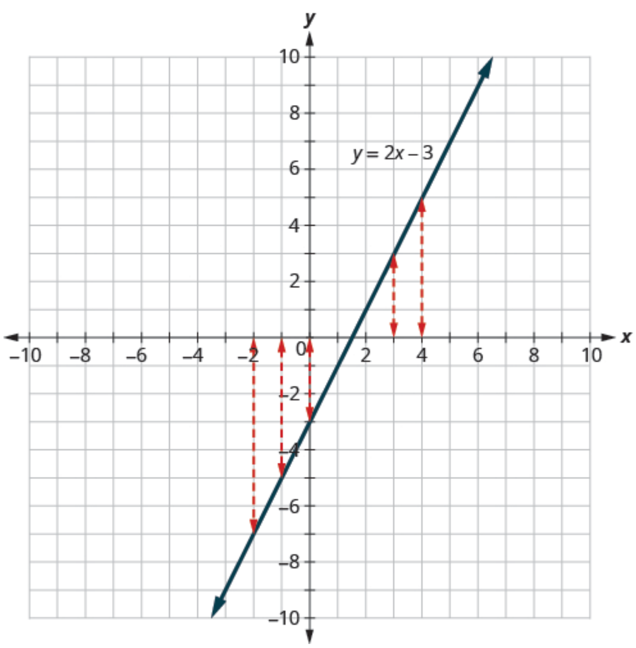
Use the Vertical Line Test

In the last section we learned how to determine if a relation is a function. The relations we looked at were expressed as a set of ordered pairs, a mapping or an equation. We will now look at how to tell if a graph is that of a function.

An ordered pair (x, y) is a solution of a linear equation, if the equation is a true statement when the *x*- and *y*-values of the ordered pair are substituted into the equation.

The graph of a linear equation is a straight line where every point on the line is a solution of the equation and every solution of this equation is a point on this line.

In Figure 3.14, we can see that, in graph of the equation $y = 2x - 3$, for every *x*-value there is only one *y*-value, as shown in the accompanying table.



$y = 2x - 3$		
x	*y*	(*x*, *y*)
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Figure 3.14

A relation is a function if every element of the domain has exactly one value in the range. So the relation defined by the equation $y = 2x - 3$ is a function.

If we look at the graph, each vertical dashed line only intersects the line at one point. This makes sense as in a function, for every *x*-value there is only one *y*-value.

If the vertical line hit the graph twice, the *x*-value would be mapped to two *y*-values, and so the graph would not represent a function.

This leads us to the vertical line test. A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point. If any vertical line intersects the graph in more than one point, the graph does not represent a function.

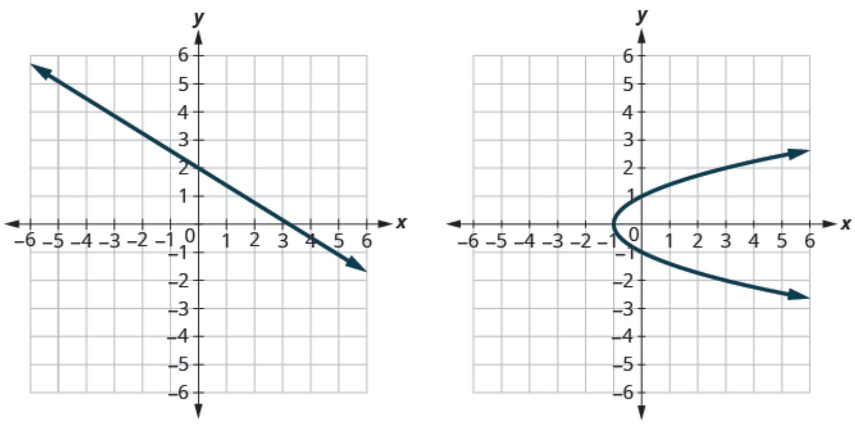
Vertical Line Test

A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.

If any vertical line intersects the graph in more than one point, the graph does not represent a function.

EXAMPLE 3.51

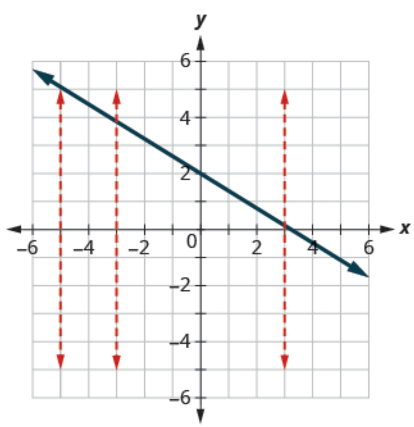
Determine whether each graph is the graph of a function.



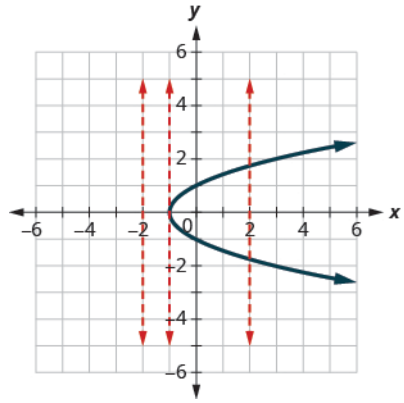
(a) (b)

Solution

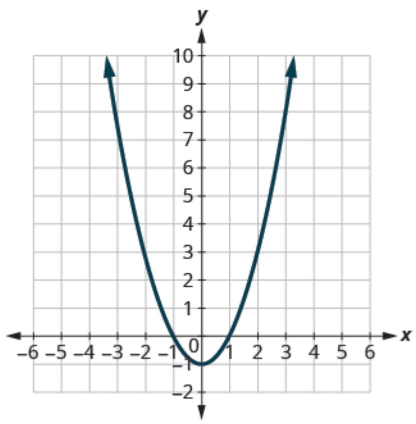
ⓐ Since any vertical line intersects the graph in at most one point, the graph is the graph of a function.



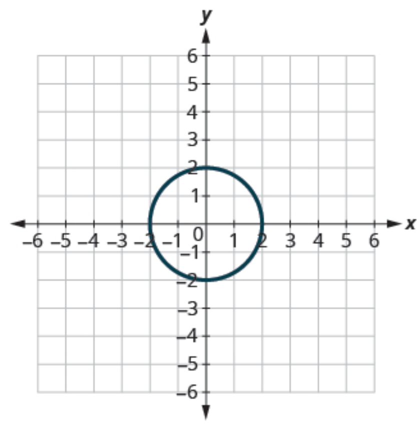
ⓑ One of the vertical lines shown on the graph, intersects it in two points. This graph does not represent a function.



> **TRY IT : :** 3.101 Determine whether each graph is the graph of a function.

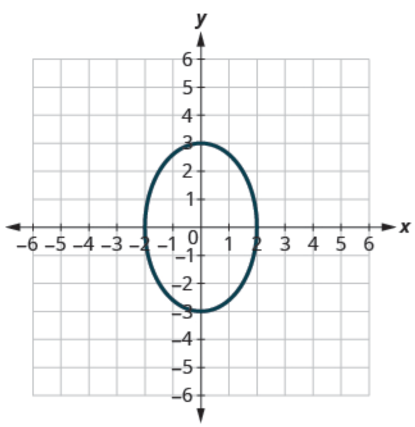


(a)

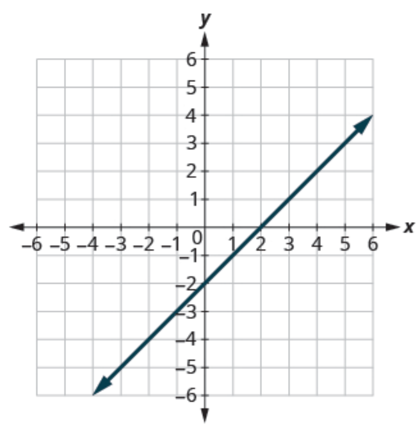


(b)

> **TRY IT : :** 3.102 Determine whether each graph is the graph of a function.



(a)



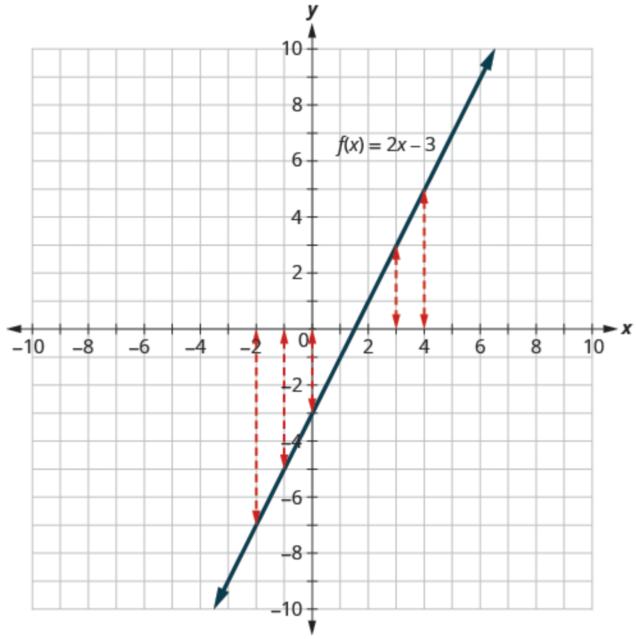
(b)

Identify Graphs of Basic Functions

We used the equation $y = 2x - 3$ and its graph as we developed the vertical line test. We said that the relation defined by the equation $y = 2x - 3$ is a function.

We can write this as in function notation as $f(x) = 2x - 3.$ It still means the same thing. The graph of the function is the graph of all ordered pairs (x, y) where $y = f(x).$ So we can write the ordered pairs as $(x, f(x)).$ It looks different but the graph will be the same.

Compare the graph of $y = 2x - 3$ previously shown in **Figure 3.14** with the graph of $f(x) = 2x - 3$ shown in **Figure 3.15**. Nothing has changed but the notation.



$f(x) = 2x - 3$		
x	$f(x)$	$(x, f(x))$
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
3	3	(3, 3)
4	5	(4, 5)

Figure 3.15

Graph of a Function

The graph of a function is the graph of all its ordered pairs, (x, y) or using function notation, $(x, f(x))$ where $y = f(x)$.

$$\begin{array}{rl} f & \text{name of function} \\ x & x\text{-coordinate of the ordered pair} \\ f(x) & y\text{-coordinate of the ordered pair} \end{array}$$

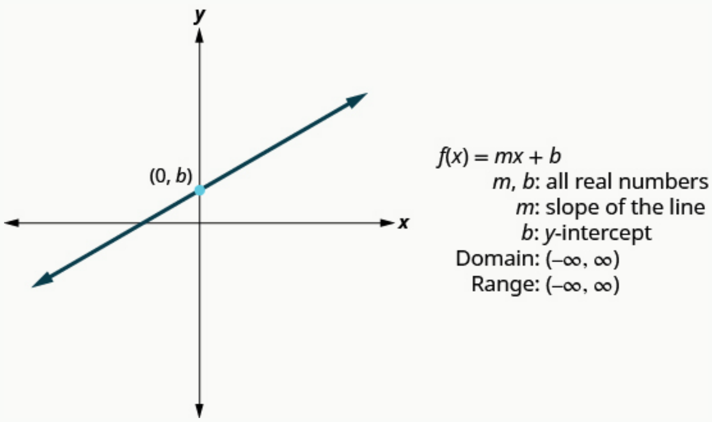
As we move forward in our study, it is helpful to be familiar with the graphs of several basic functions and be able to identify them.

Through our earlier work, we are familiar with the graphs of linear equations. The process we used to decide if $y = 2x - 3$ is a function would apply to all linear equations. All non-vertical linear equations are functions. Vertical lines are not functions as the *x*-value has infinitely many *y*-values.

We wrote linear equations in several forms, but it will be most helpful for us here to use the slope-intercept form of the linear equation. The slope-intercept form of a linear equation is $y = mx + b$. In function notation, this linear function becomes $f(x) = mx + b$ where *m* is the slope of the line and *b* is the *y*-intercept.

The domain is the set of all real numbers, and the range is also the set of all real numbers.

Linear Function

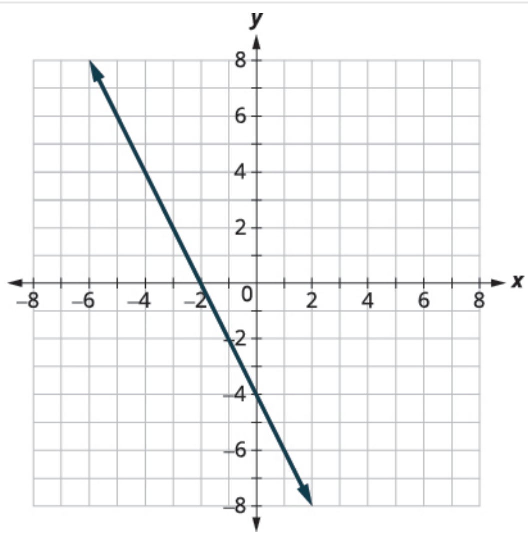


We will use the graphing techniques we used earlier, to graph the basic functions.

EXAMPLE 3.52

Graph: $f(x) = -2x - 4$.

Solution

	$f(x) = -2x - 4$
We recognize this as a linear function.	
Find the slope and y-intercept.	$m = -2$ $b = -4$
Graph using the slope intercept.	

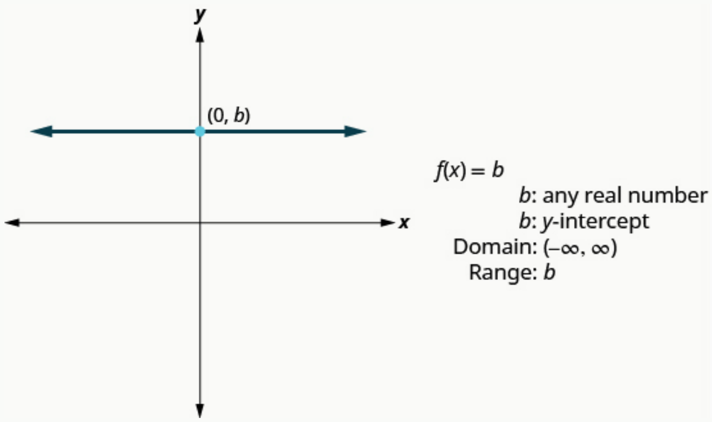
TRY IT : : 3.103 Graph: $f(x) = -3x - 1$

TRY IT : : 3.104 Graph: $f(x) = -4x - 5$

The next function whose graph we will look at is called the constant function and its equation is of the form $f(x) = b$, where *b* is any real number. If we replace the $f(x)$ with y, we get $y = b.$ We recognize this as the horizontal line whose *y*-intercept is *b*. The graph of the function $f(x) = b,$ is also the horizontal line whose *y*-intercept is *b*.

Notice that for any real number we put in the function, the function value will be *b*. This tells us the range has only one value, *b*.

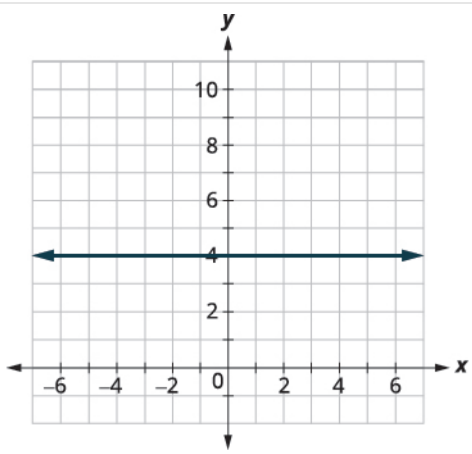
Constant Function



EXAMPLE 3.53

Graph: $f(x) = 4.$

Solution

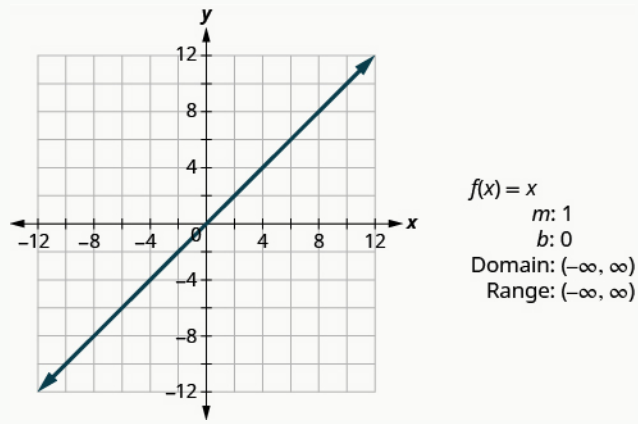
	$f(x) = 4$
We recognize this as a constant function.	
The graph will be a horizontal line through $(0, 4).$	

TRY IT : : 3.105 Graph: $f(x) = -2.$

TRY IT : : 3.106 Graph: $f(x) = 3.$

The identity function, $f(x) = x$ is a special case of the linear function. If we write it in linear function form, $f(x) = 1x + 0,$ we see the slope is 1 and the *y*-intercept is 0.

Identity Function



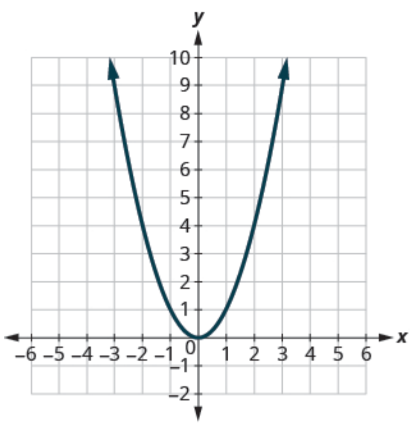
The next function we will look at is not a linear function. So the graph will not be a line. The only method we have to graph this function is point plotting. Because this is an unfamiliar function, we make sure to choose several positive and negative values as well as 0 for our x-values.

EXAMPLE 3.54

Graph: $f(x) = x^2$.

Solution

We choose x-values. We substitute them in and then create a chart as shown.



x	$f(x) = x^2$	$(x, f(x))$
−3	9	(−3, 9)
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

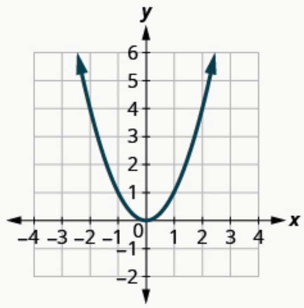
TRY IT : : 3.107 Graph: $f(x) = x^2$.

TRY IT : : 3.108 $f(x) = -x^2$

Looking at the result in Example 3.54, we can summarize the features of the square function. We call this graph a parabola. As we consider the domain, notice any real number can be used as an x-value. The domain is all real numbers.

The range is not all real numbers. Notice the graph consists of values of y never go below zero. This makes sense as the square of any number cannot be negative. So, the range of the square function is all non-negative real numbers.

Square Function



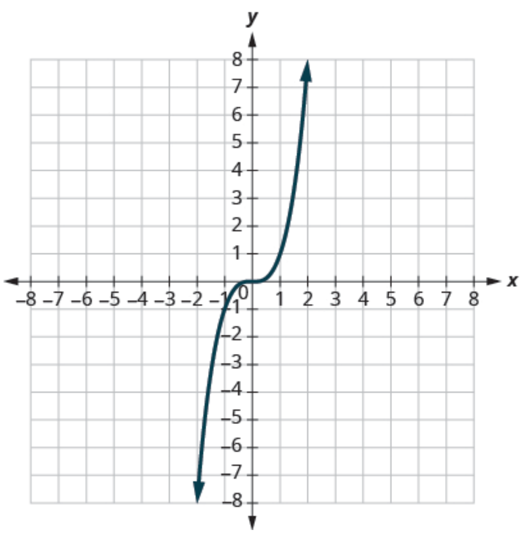
The next function we will look at is also not a linear function so the graph will not be a line. Again we will use point plotting, and make sure to choose several positive and negative values as well as 0 for our *x*-values.

EXAMPLE 3.55

Graph: $f(x) = x^3$.

Solution

We choose *x*-values. We substitute them in and then create a chart.



x	$f(x) = x^3$	$(x, f(x))$
–2	–8	(–2, –8)
–1	–1	(–1, –1)
0	0	(0, 0)
1	1	(1, 1)
2	8	(2, 8)

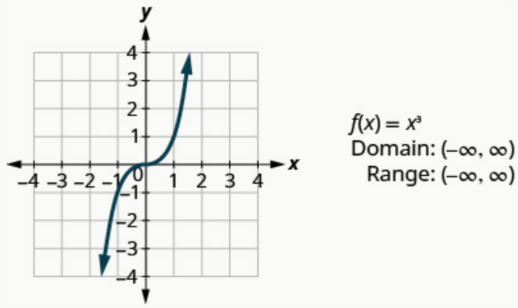
TRY IT : : 3.109 Graph: $f(x) = x^3$.

TRY IT : : 3.110 Graph: $f(x) = -x^3$.

Looking at the result in **Example 3.55**, we can summarize the features of the cube function. As we consider the domain, notice any real number can be used as an *x*-value. The domain is all real numbers.

The range is all real numbers. This makes sense as the cube of any non-zero number can be positive or negative. So, the range of the cube function is all real numbers.

Cube Function



The next function we will look at does not square or cube the input values, but rather takes the square root of those values.

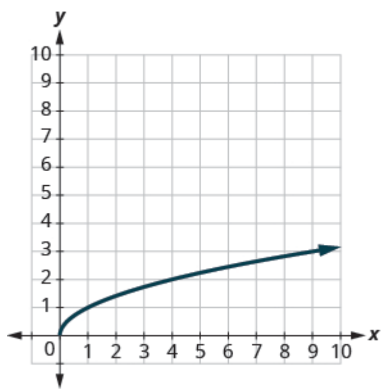
Let's graph the function $f(x) = \sqrt{x}$ and then summarize the features of the function. Remember, we can only take the square root of non-negative real numbers, so our domain will be the non-negative real numbers.

EXAMPLE 3.56

$f(x) = \sqrt{x}$

Solution

We choose x-values. Since we will be taking the square root, we choose numbers that are perfect squares, to make our work easier. We substitute them in and then create a chart.

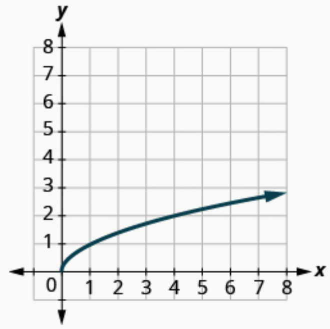


x	$f(x) = \sqrt{x}$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)

TRY IT : : 3.111 Graph: $f(x) = \sqrt{x}$.

TRY IT : : 3.112 Graph: $f(x) = -\sqrt{x}$.

Square Root Function



$f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$

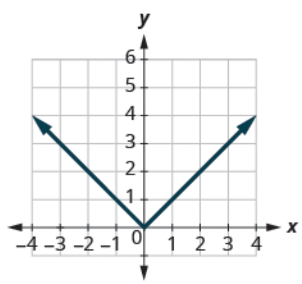
Our last basic function is the absolute value function, $f(x) = |x|$. Keep in mind that the absolute value of a number is its distance from zero. Since we never measure distance as a negative number, we will never get a negative number in the range.

EXAMPLE 3.57

Graph: $f(x) = |x|$.

Solution

We choose x-values. We substitute them in and then create a chart.

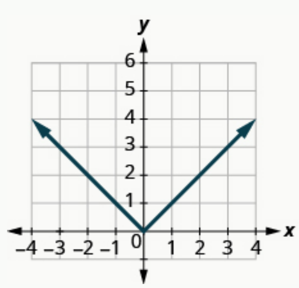


x	$f(x) = \lvert x \rvert$	$(x, f(x))$
−3	3	(−3, 3)
−2	2	(−2, 2)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

TRY IT : : 3.113 Graph: $f(x) = |x|$.

TRY IT : : 3.114 Graph: $f(x) = -|x|$.

Absolute Value Function



$f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Read Information from a Graph of a Function

In the sciences and business, data is often collected and then graphed. The graph is analyzed, information is obtained from the graph and then often predictions are made from the data.

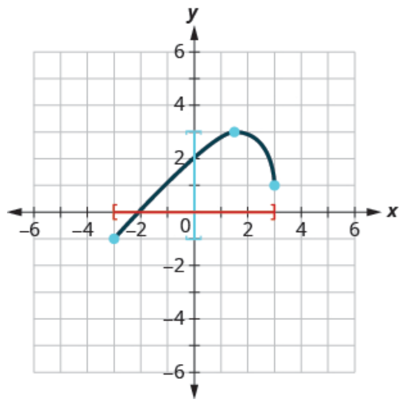
We will start by reading the domain and range of a function from its graph.

Remember the domain is the set of all the *x*-values in the ordered pairs in the function. To find the domain we look at the graph and find all the values of *x* that have a corresponding value on the graph. Follow the value *x* up or down vertically. If you hit the graph of the function then *x* is in the domain.

Remember the range is the set of all the *y*-values in the ordered pairs in the function. To find the range we look at the graph and find all the values of *y* that have a corresponding value on the graph. Follow the value *y* left or right horizontally. If you hit the graph of the function then *y* is in the range.

EXAMPLE 3.58

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



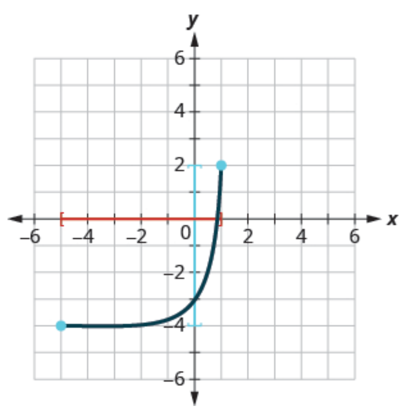
Solution

To find the domain we look at the graph and find all the values of *x* that correspond to a point on the graph. The domain is highlighted in red on the graph. The domain is $[-3,\ 3]$.

To find the range we look at the graph and find all the values of *y* that correspond to a point on the graph. The range is highlighted in blue on the graph. The range is $[-1,\ 3]$.

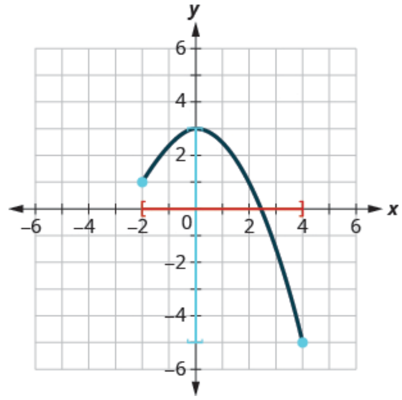
TRY IT : : 3.115

Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



TRY IT : : 3.116

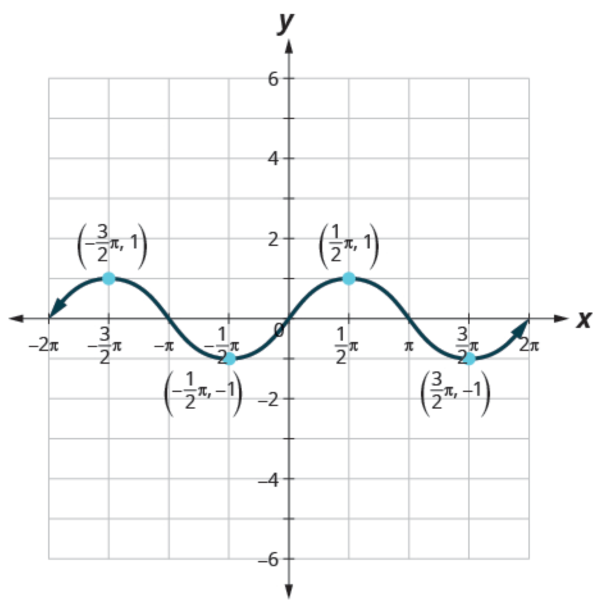
Use the graph of the function to find its domain and range. Write the domain and range in interval notation.



We are now going to read information from the graph that you may see in future math classes.

EXAMPLE 3.59

Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.

ⓑ Find: $f\left(\frac{3}{2}\pi\right)$.

ⓒ Find: $f\left(-\frac{1}{2}\pi\right)$.

ⓓ Find the values for *x* when $f(x) = 0$.

ⓔ Find the *x*-intercepts.

ⓕ Find the *y*-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation.

Solution

ⓐ When $x = 0$, the function crosses the *y*-axis at 0. So, $f(0) = 0$.

ⓑ When $x = \frac{3}{2}\pi$, the *y*-value of the function is -1. So, $f\left(\frac{3}{2}\pi\right) = -1$.

ⓒ When $x = -\frac{1}{2}\pi$, the *y*-value of the function is -1. So, $f\left(-\frac{1}{2}\pi\right) = -1$.

ⓓ The function is 0 at the points, $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$. The *x*-values when $f(x) = 0$ are $-2\pi, -\pi, 0, \pi, 2\pi$.

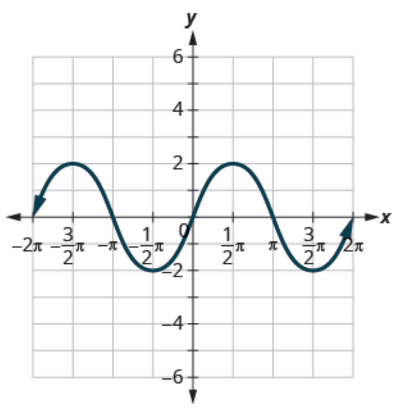
ⓔ The *x*-intercepts occur when $y = 0$. So the *x*-intercepts occur when $f(x) = 0$. The *x*-intercepts are $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0), (2\pi, 0)$.

ⓕ The *y*-intercepts occur when $x = 0$. So the *y*-intercepts occur at $f(0)$. The *y*-intercept is $(0, 0)$.

ⓖ This function has a value when *x* is from -2π to 2π. Therefore, the domain in interval notation is $[-2\pi, 2\pi]$.

ⓗ This function values, or *y*-values go from -1 to 1. Therefore, the range, in interval notation, is $[-1, 1]$.

TRY IT : : 3.117 Use the graph of the function to find the indicated values.



ⓐ Find: $f(0)$.

ⓑ Find: $f\left(\frac{1}{2}\pi\right)$.

ⓒ Find: $f\left(-\frac{3}{2}\pi\right)$.

ⓓ Find the values for *x* when $f(x) = 0$.

ⓔ Find the *x*-intercepts.

ⓕ Find the *y*-intercepts.

ⓖ Find the domain. Write it in interval notation.

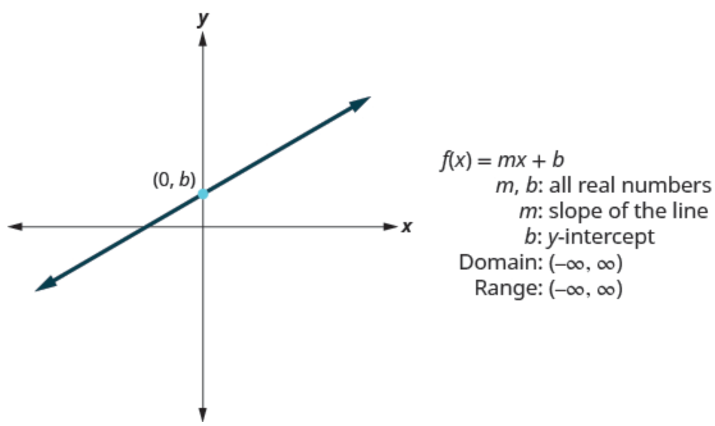
ⓗ Find the range. Write it in interval notation.

Summary of Graphs of Functions

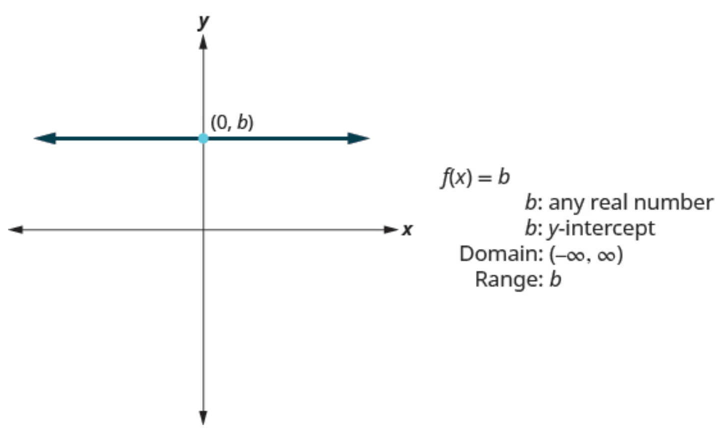
- **Vertical Line Test**
  - A set of points in a rectangular coordinate system is the graph of a function if every vertical line intersects the graph in at most one point.
  - If any vertical line intersects the graph in more than one point, the graph does not represent a function.
- **Graph of a Function**
  - The graph of a function is the graph of all its ordered pairs, (x, y) or using function notation, $(x, f(x))$ where $y = f(x)$.

f	name of function
x	x-coordinate of the ordered pair
$f(x)$	y-coordinate of the ordered pair

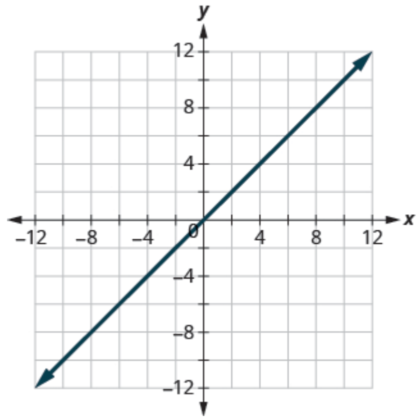
- **Linear Function**



- **Constant Function**

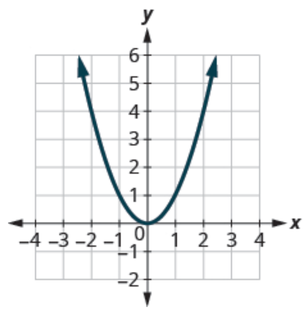


- **Identity Function**



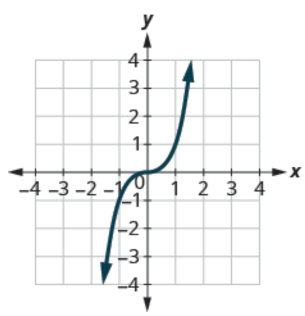
$f(x) = x$
m: 1
b: 0
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Square Function**



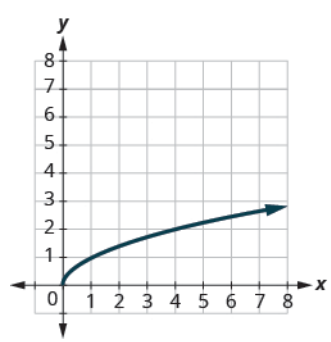
$f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

- **Cube Function**



$f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

- **Square Root Function**



$f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$

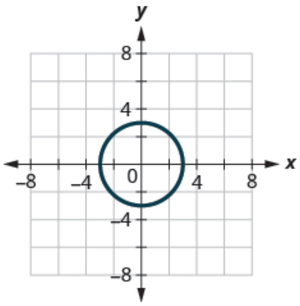
3.6 EXERCISES

Practice Makes Perfect

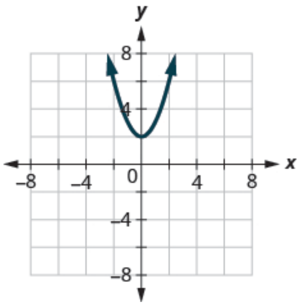
Use the Vertical Line Test

In the following exercises, determine whether each graph is the graph of a function.

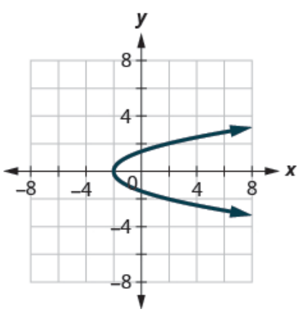
337. ⓐ



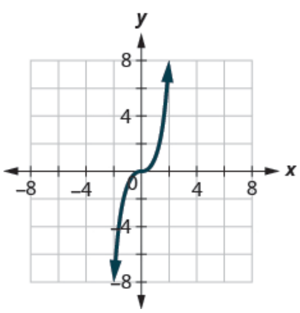
ⓑ



339. ⓐ



ⓑ



Identify Graphs of Basic Functions

In the following exercises, ⓐ graph each function ⓑ state its domain and range. Write the domain and range in interval notation.

341. $f(x) = 3x + 4$

343. $f(x) = -x - 2$

345. $f(x) = -2x + 2$

347. $f(x) = \frac{1}{2}x + 1$

351. $f(x) = -3$

351. $f(x) = -3$

355. $f(x) = -2x$

357. $f(x) = 3x^2$

363. $f(x) = x^2 - 1$

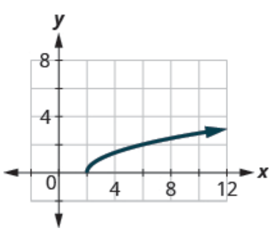
365. $f(x) = -2x^3$

373. $f(x) = 3|x|$

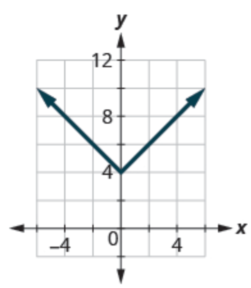
Read Information from a Graph of a Function

In the following exercises, use the graph of the function to find its domain and range. Write the domain and range in interval notation.

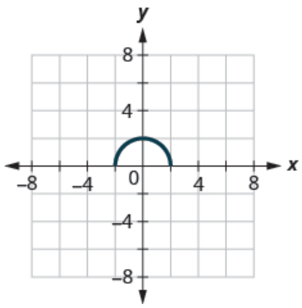
377.



379.

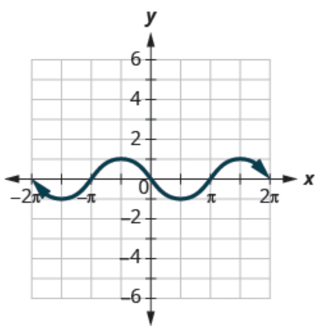


381.



In the following exercises, use the graph of the function to find the indicated values.

383.



ⓐ Find: $f(0)$.

ⓑ Find: $f\left(\frac{1}{2}\pi\right)$.

ⓒ Find: $f\left(-\frac{3}{2}\pi\right)$.

ⓓ Find the values for *x* when $f(x) = 0$.

ⓔ Find the *x*-intercepts.

ⓕ Find the *y*-intercepts.

ⓖ Find the domain. Write it in interval notation.

ⓗ Find the range. Write it in interval notation.