SUPPLEMENT 1C

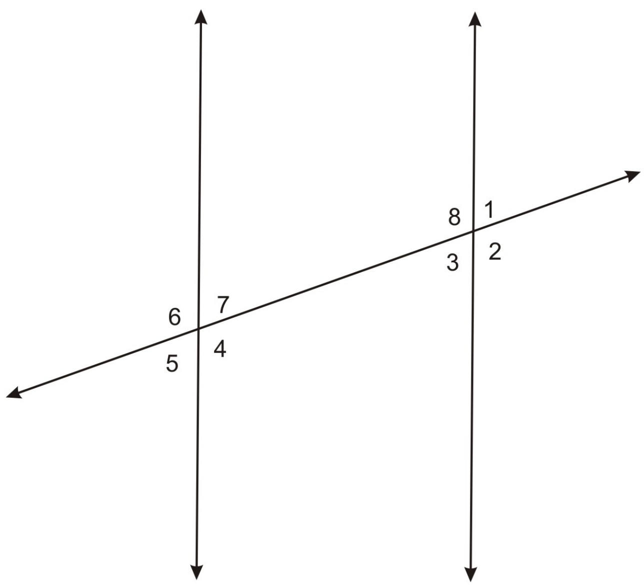
1.4 Parallel Lines and Transversals

Review of Terms

As a quick review, it is helpful to practice identifying different categories of angles.

Example 1

In the diagram below, two vertical parallel lines are cut by a transversal.



Identify the pairs of corresponding angles, alternate interior angles, alternate exterior angles, and consecutive interior angles.

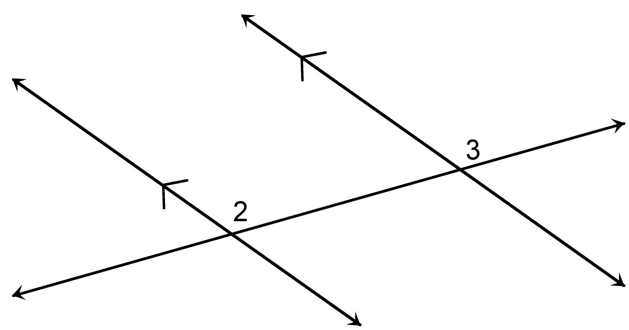
- Corresponding angles: Corresponding angles are formed on different lines, but in the same relative position to the transversal—in other words, they face the same direction. There are four pairs of corresponding angles in this diagram—$\angle 6$ and $\angle 8, \angle 7$ and $\angle 1, \angle 5$ and $\angle 3$, and $\angle 4$ and $\angle 2$.
- Alternate interior angles: These angles are on the interior of the lines crossed by the transversal and are on opposite sides of the transversal. There are two pairs of alternate interior angles in this diagram—$\angle 7$ and $\angle 3$, and $\angle 8$ and $\angle 4$.
- Alternate exterior angles: These are on the exterior of the lines crossed by the transversal and are on opposite sides of the transversal. There are two pairs of alternate exterior angles in this diagram—$\angle 1$ and $\angle 5$, and $\angle 2$ and $\angle 6$.

- Consecutive interior angles: Consecutive interior angles are in the interior region of the lines crossed by the transversal, and are on the same side of the transversal. There are two pairs of consecutive interior angles in this diagram—$\angle 7$ and $\angle 8$ and $\angle 3$ and $\angle 4$.

Corresponding Angles Postulate

By now you have had lots of practice and should be able to easily identify relationships between angles.

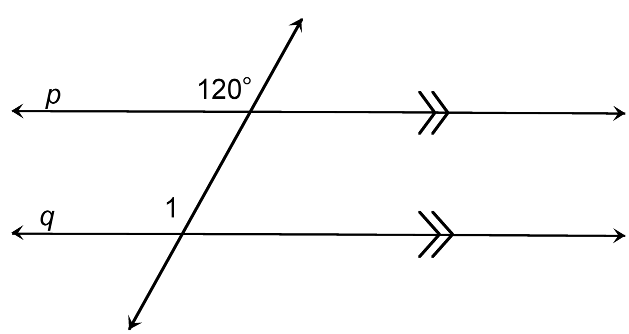
Corresponding Angles Postulate: If the lines crossed by a transversal are parallel, then corresponding angles will be congruent. Examine the following diagram.



You already know that $\angle 2$ and $\angle 3$ are corresponding angles because they are formed by two lines crossed by a transversal and have the same relative placement next to the transversal. The Corresponding Angles postulate says that *because the lines are parallel to each other,* the corresponding *angles will be congruent.*

Example 2

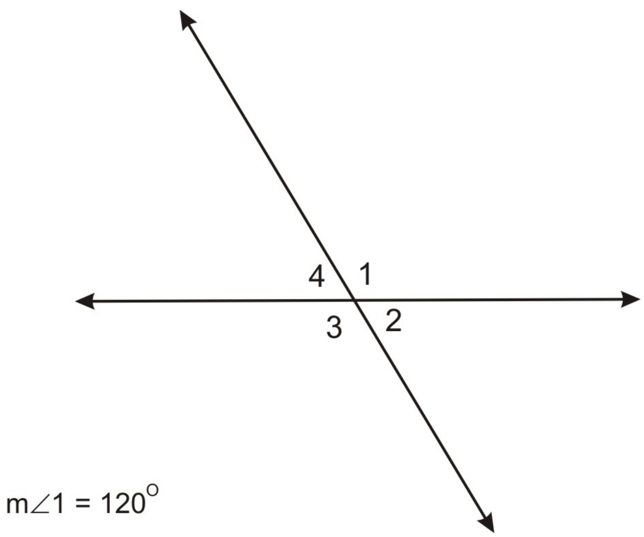
In the diagram below, lines p and q are parallel. What is the measure of $\angle 1$*?*



Because lines p and q are parallel, the 120° angle and $\angle 1$ are corresponding angles, we know by the Corresponding Angles Postulate that they are congruent. Therefore, $m\angle 1 = 120^\circ$.

Alternate Interior Angles Theorem

Now that you know the Corresponding Angles Postulate, you can use it to derive the relationships between all other angles formed when two lines are crossed by a transversal. Examine the angles formed below.



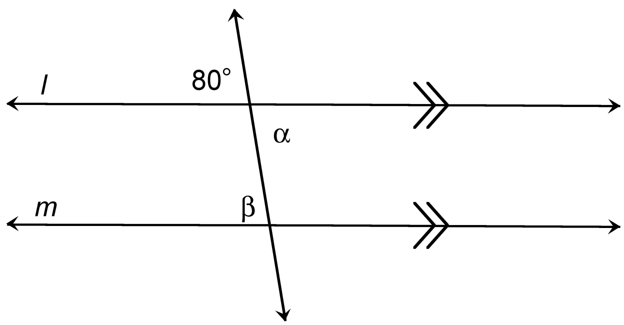
If you know that the measure of $\angle 1$ is 120°, you can find the measurement of all the other angles. For example, $\angle 1$ and $\angle 2$ must be **supplementary** (sum to 180°) because together they are a linear pair (we are using the Linear Pair Postulate here). So, to find $m\angle 2$, subtract 120° from 180°.

$$m\angle 2 = 180^\circ - 120^\circ$$
$$m\angle 2 = 60^\circ$$

So, $m\angle 2 = 60^\circ$. Knowing that $\angle 2$ and $\angle 3$ are also supplementary means that $m\angle 3 = 120^\circ$, since $120 + 60 = 180$. If $m\angle 3 = 120^\circ$, then $m\angle 4$ must be 60°, because $\angle 3$ and $\angle 4$ are also supplementary. Notice that $\angle 1 \cong \angle 3$ (they both measure 120°) and $\angle 2 \cong \angle 4$ (both measure 60°). These angles are called **vertical angles**. Vertical angles are on opposite sides of intersecting lines, and will always be congruent by the Vertical Angles Theorem, which we proved in an earlier chapter. Using this information, you can now deduce the relationship between alternate interior angles.

Example 3

Lines l and m in the diagram below are parallel. What are the measures of angles α and β?



In this problem, you need to find the angle measures of two alternate interior angles given an exterior angle. Use what you know. There is one angle that measures 80°. Angle β corresponds to the 80° angle. So by the Corresponding Angles Postulate, $m\angle\beta = 80^\circ$.

Now, because $\angle\alpha$ is made by the same intersecting lines and is opposite the 80° angle, these two angles are vertical angles. Since you already learned that vertical angles are congruent, we conclude $m\angle\alpha = 80^\circ$. Finally, compare angles α and β. They both measure 80°, so they are congruent. This will be true any time two parallel lines are cut by a transversal.

We have shown that alternate interior angles are congruent in this example. Now we need to show that it is always true for any angles.

Alternate Interior Angles Theorem

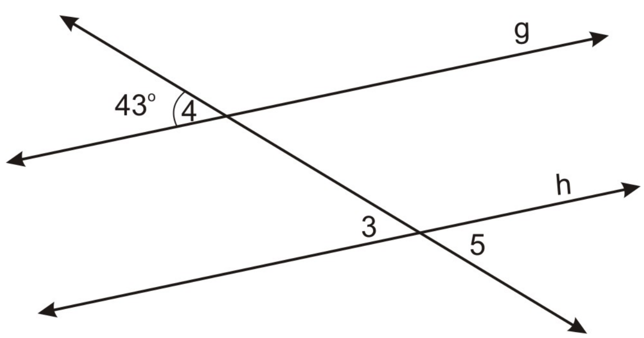
Alternate interior angles formed by two parallel lines and a transversal will always be congruent.

Alternate Exterior Angles Theorem

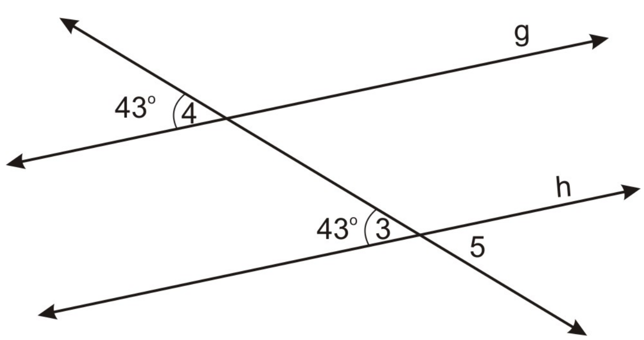
Now you know that pairs of corresponding, vertical, and alternate interior angles are congruent. We will use logic to show that Alternate Exterior Angles are congruent—when two parallel lines are crossed by a transversal, of course.

Example 4

Lines g and h in the diagram below are parallel. If $m\angle 4 = 43^\circ$, what is the measure of $\angle 5$?



You know from the problem that $m\angle 4 = 43^\circ$. That means that $\angle 4's$ corresponding angle, which is $\angle 3$, will measure 43° as well.



The corresponding angle you just filled in is also vertical to $\angle 5$. Since vertical angles are congruent, you can conclude $m\angle 5 = 43^\circ$.

Alternate Exterior Angles Theorem

If two parallel lines are crossed by a transversal, then alternate exterior angles are congruent.

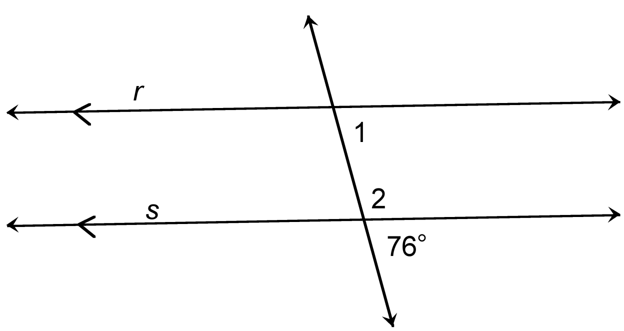
We omit the proof here, but note that you can prove alternate exterior angles are congruent by following the method of example 4, but not using any particular measures for the angles.

Consecutive Interior Angles Theorem

The last category of angles to explore in this lesson is consecutive interior angles. They fall on the interior of the parallel lines and are on the same side of the transversal. Use your knowledge of corresponding angles to identify their mathematical relationship.

Example 5

Lines r and s in the diagram below are parallel. If the angle corresponding to $\angle 1$ measures 76°, what is $m\angle 2$?



This process should now seem familiar. The given 76° angle is adjacent to $\angle 2$ and they form a linear pair. Therefore, the angles are supplementary. So, to find $m\angle 2$, subtract 76° from 180°.

$$m\angle 2 = 180 - 76$$
$$m\angle 2 = 104^\circ$$

This example shows that if two parallel lines are cut by a transversal, the consecutive interior angles are supplementary; they sum to 180°. This is called the Consecutive Interior Angles Theorem. We restate it here for clarity.

Consecutive Interior Angles Theorem

If two parallel lines are crossed by a transversal, then consecutive interior angles are supplementary.

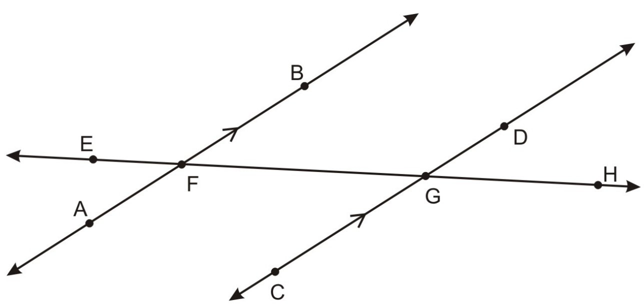
Points To Consider

You used logic to work through a number of different scenarios in this lesson. Always apply logic to mathematical situations to make sure that they are reasonable. Even if it doesn't help you solve the problem, it will help you notice careless errors or other mistakes.

Review Questions

Solve each problem.

Use the diagram below for Questions 1-4. In the diagram, lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.

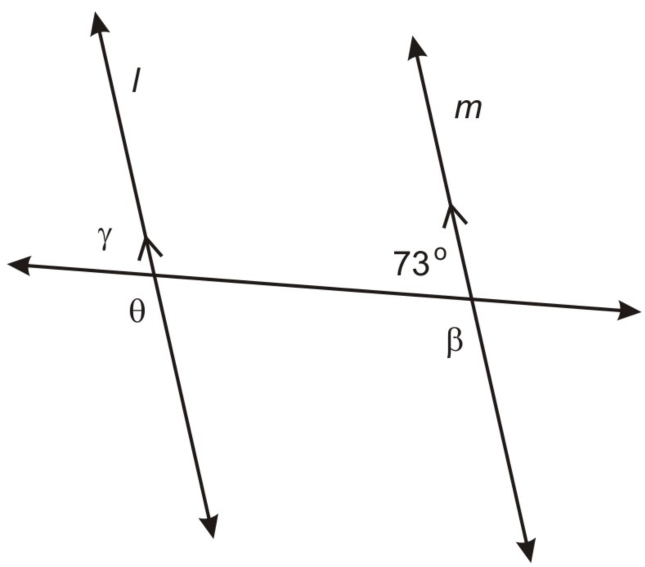


1. What term best describes the relationship between $\angle AFG$ and $\angle CGH$?

   (a) alternate exterior angles
   (b) consecutive interior angles
   (c) corresponding angles
   (d) alternate interior angles

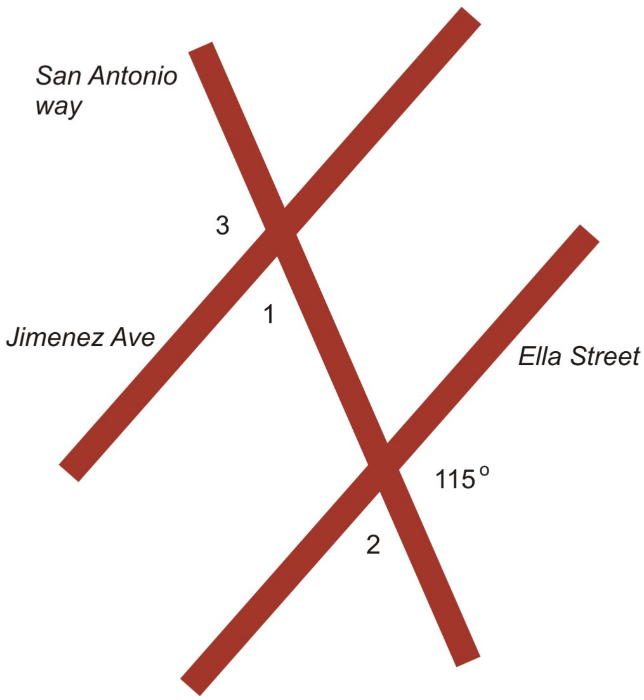
2. What term best describes the mathematical relationship between $\angle BFG$ and $\angle DGF$?
   (a) congruent
   (b) supplementary
   (c) complementary
   (d) no relationship
3. What term best describes the relationship between $\angle FGD$ and $\angle AFG$?
   (a) alternate exterior angles
   (b) consecutive interior angles
   (c) complementary
   (d) alternate interior angles
4. What term best describes the mathematical relationship between $\angle AFE$ and $\angle CGH$?
   (a) congruent
   (b) supplementary
   (c) complementary
   (d) no relationship

   Use the diagram below for questions 5-7. In the diagram, lines l and m are parallel γ, β, θ represent the measures of the angles.



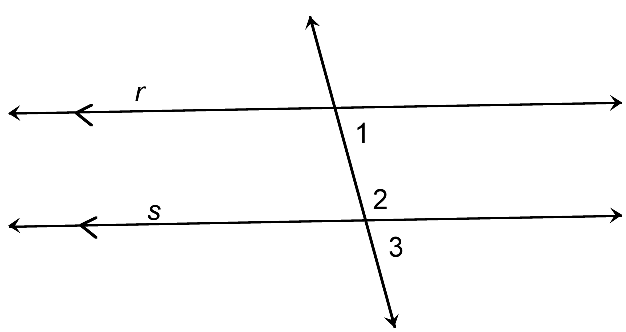
5. What is γ?
6. What is β?
7. What is θ?

The map below shows some of the streets in Ahmed's town.



Jimenez Ave and Ella Street are parallel. Use this map to answer questions 8-10.

8. What is the measure of angle 1?
9. What is the measure of angle 2?
10. What is the measure of angle 3?
11. Prove the Consecutive Interior Angle Theorem. Given $r||s$, prove $\angle 1$ and $\angle 2$ are supplementary.



Review Answers

1. c
2. a
3. d
4. b
5. 73°
6. 107°
7. 107°
8. 65°
9. 65°
10. 115°

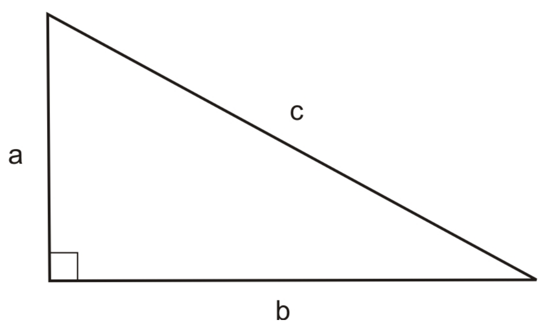
1.5 The Pythagorean Theorem

Learning Objectives

- Identify and employ the Pythagorean Theorem when working with right triangles.
- Identify common Pythagorean triples.
- Use the Pythagorean Theorem to find the area of isosceles triangles.
- Use the Pythagorean Theorem to derive the distance formula on a coordinate grid.

Introduction

The triangle below is a right triangle.



The sides labeled a and b are called the **legs** of the triangle and they meet at the right angle. The third side, labeled c is called the **hypotenuse.** The hypotenuse is opposite the right angle. The hypotenuse of a right triangle is also the longest side.

The Pythagorean Theorem states that the length of the hypotenuse squared will equal the sum of the squares of the lengths of the two legs. In the triangle above, the sum of the squares of the legs is $a^2 + b^2$ and the square of the hypotenuse is c^2.

The Pythagorean Theorem: Given a right triangle with legs whose lengths are a and b and a hypotenuse of length c,

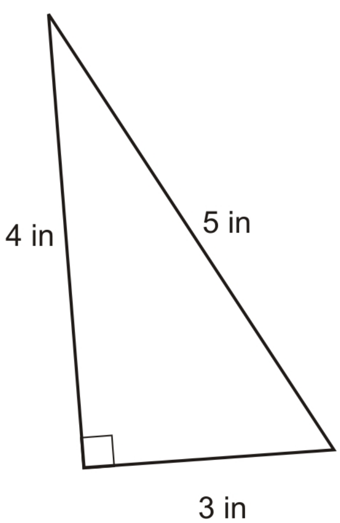
$$a^2 + b^2 = c^2$$

Be careful when using this theorem—you must make sure that the legs are labeled a and b and the hypotenuse is labeled c to use this equation. A more accurate way to write the Pythagorean Theorem is:

$$(\text{leg}_1)^2 + (\text{leg}_2)^2 = \text{hypotenuse}^2$$

Example 1

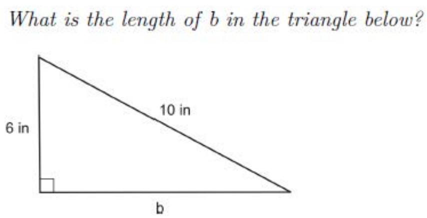
Use the side lengths of the following triangle to test the Pythagorean Theorem.



The legs of the triangle above are 3 inches and 4 inches . The hypotenuse is 5 inches . So, $a = 3, b = 4$, and $c = 5$. We can substitute these values into the formula for the Pythagorean Theorem to verify that the relationship works:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \\ 25 &= 25 \end{aligned}$$

Use the Pythagorean Theorem to find the length of the missing leg, b. Set up the equation $a^2 + b^2 = c^2$, letting $a = 6$ and $b = 10$. Be sure to simplify the exponents and roots carefully, remember to use inverse operations to solve the equation, and always keep both sides of the equation balanced.

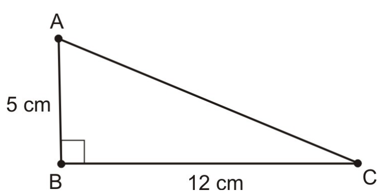


$$\begin{aligned}
a^2 + b^2 &= c^2 \\
6^2 + b^2 &= 10^2 \\
36 + b^2 &= 100 \\
36 + b^2 - 36 &= 100 - 36 \\
b^2 &= 64 \\
\sqrt{b^2} &= \sqrt{64} \\
b &= \pm 8 \\
b &= 8
\end{aligned}$$

In algebra you learned that $\sqrt{x^2} = \pm x$ because, for example, $(5)^2 = (-5)^2 = 25$. However, in this case (and in much of geometry), we are only interested in the positive solution to $b = \sqrt{64}$ because geometric lengths are positive. So, in example 2, we can disregard the solution $b = -8$, and our final answer is $b = 8$ inches.

Example 3

Find the length of the missing side in the triangle below.



Use the Pythagorean Theorem to set up an equation and solve for the missing side. Let $a = 5$ and $b = 12$.

$$\begin{aligned}
a^2 + b^2 &= c^2 \\
5^2 + 12^2 &= c^2 \\
25 + 144 &= c^2 \\
169 &= c^2 \\
\sqrt{169} &= \sqrt{c^2} \\
13 &= c
\end{aligned}$$

So, the length of the missing side is 13 centimeters.

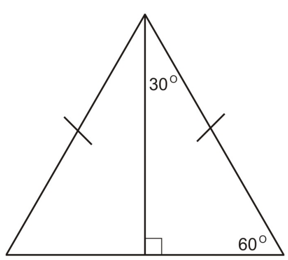
1.6 Special Right Triangles

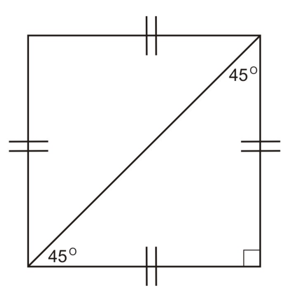
Learning Objectives

- Identify and use the ratios involved with right isosceles triangles.
- Identify and use the ratios involved with $30^\circ - 60^\circ - 90^\circ$ triangles.
- Identify and use ratios involved with equilateral triangles.
- Employ right triangle ratios when solving real-world problems.

Introduction

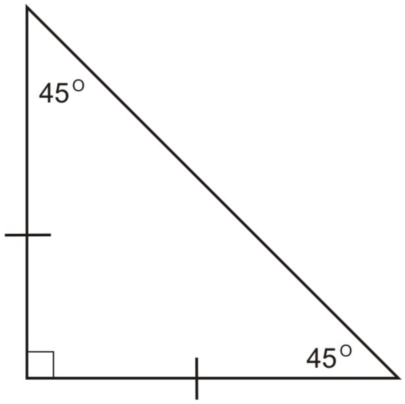
What happens when you cut an equilateral triangle in half using an altitude? You get two right triangles. What about a square? If you draw a diagonal across a square you also get two right triangles. These two right triangles are *special special right triangles* called the $30^\circ - 60^\circ - 90^\circ$ and the $45^\circ - 45^\circ - 90^\circ$ right triangles. They have unique properties and if you understand the relationships between the sides and angles in these triangles, you will do well in geometry, trigonometry, and beyond.





Right Isosceles Triangles

The first type of right triangle to examine is **isosceles.** As you know, isosceles triangles have two sides that are the same length. Additionally, the base angles of an isosceles triangle are congruent as well. An isosceles **right** triangle will always have base angles that each measure 45° and a vertex angle that measures 90°.

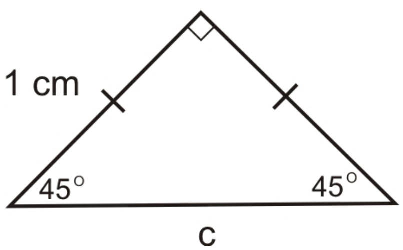


Don't forget that the base angles are the angles across from the congruent sides. They don't have to be on the bottom of the figure.

Because the angles of all $45^\circ - 45^\circ - 90^\circ$ triangles will, by definition, remain the same, all $45^\circ - 45^\circ - 90^\circ$ triangles are **similar,** so their sides will always be proportional. To find the relationship between the sides, use the Pythagorean Theorem.

Example 1

The isosceles right triangle below has legs measuring 1 centimeter.



Use the Pythagorean Theorem to find the length of the hypotenuse.

Since the legs are 1 centimeter each, substitute 1 for both a and b, and solve for c:

$$\begin{aligned}
a^2 + b^2 &= c^2 \\
1^2 + 1^2 &= c^2 \\
1 + 1 &= c^2 \\
2 &= c^2 \\
\sqrt{2} &= \sqrt{c^2} \\
c &= \sqrt{2}
\end{aligned}$$

In this example $c = \sqrt{2}$ cm.

What if each leg in the example above was 5 cm? Then we would have

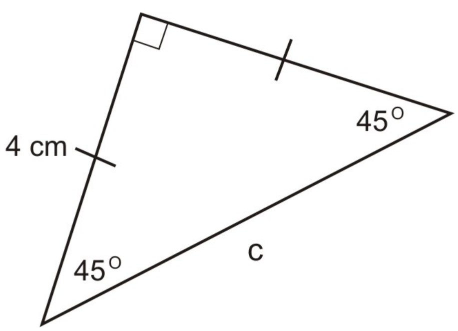
$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
5^2 + 5^2 &= c^2 \\
25 + 25 &= c^2 \\
50 &= c^2 \\
\sqrt{50} &= \sqrt{c^2} \\
c &= 5\sqrt{2}
\end{aligned}
$$

If each leg is 5 cm, then the hypotenuse is $5\sqrt{2}$ cm.

When the length of each leg was 1, the hypotenuse was $1\sqrt{2}$. When the length of each leg was 5, the hypotenuse was $5\sqrt{2}$. Is this a coincidence? No. Recall that the legs of all $45^\circ - 45^\circ - 90^\circ$ triangles are proportional. The hypotenuse of an isosceles right triangle will always equal the product of the length of one leg and $\sqrt{2}$. Use this information to solve the problem in example 2.

Example 2

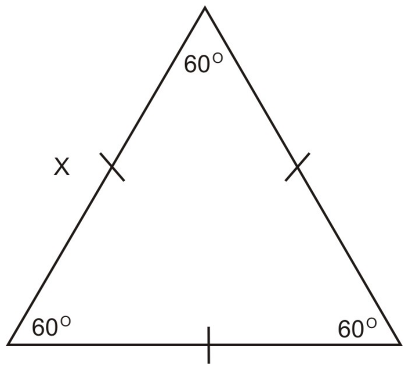
What is the length of the hypotenuse in the triangle below?



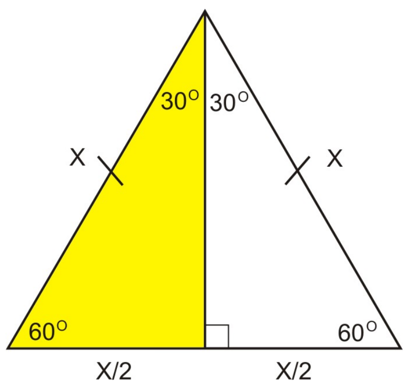
Since the length of the hypotenuse is the product of one leg and $\sqrt{2}$, you can easily calculate this length. One leg is 4 inches, so the hypotenuse will be $4\sqrt{2}$ inches, or about 5.66 inches.

Equilateral Triangles

Remember that an equilateral triangle has sides that all have the same length. Equilateral triangles are also equiangular—all angles have the same measure. In an equilateral triangle, all angles measure exactly 60°.



Notice what happens when you divide an equilateral triangle in half.



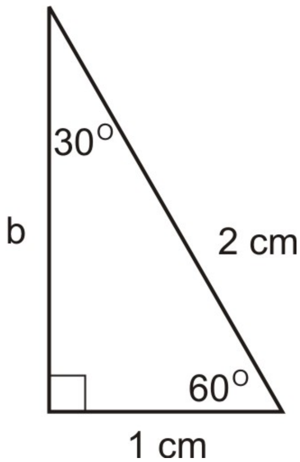
When an equilateral triangle is divided into two equal parts using an altitude, each resulting right triangle is a $30^\circ - 60^\circ - 90^\circ$ triangle. The hypotenuse of the resulting triangle was the side of the original, and the shorter leg is half of an original side. This is why the hypotenuse is always twice the length of the shorter leg in a $30^\circ - 60^\circ - 90^\circ$ triangle. You can use this information to solve problems about equilateral triangles.

30º-60º-90º Triangles

Another important type of right triangle has angles measuring 30°, 60°, and 90°. Just as you found a constant ratio between the sides of an isosceles right triangle, you can find constant ratios here as well. Use the Pythagorean Theorem to discover these important relationships.

Example 3

Find the length of the missing leg in the following triangle. Use the Pythagorean Theorem to find your answer.



Just like you did for $45^\circ - 45^\circ - 90^\circ$ triangles, use the Pythagorean theorem to find the missing side. In this diagram, you are given two measurements: the hypotenuse (c) is 2 cm and the shorter leg (a) is 1 cm. Find the length of the missing leg (b).

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + b^2 &= 2^2 \\ 1 + b^2 &= 4 \\ b^2 &= 3 \\ b &= \sqrt{3} \end{aligned}$$

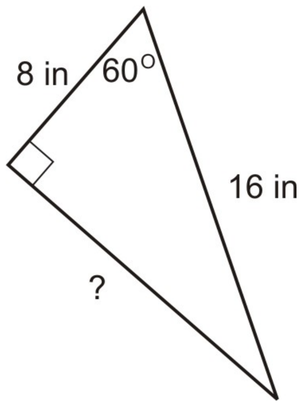
You can leave the answer in radical form as shown, or use your calculator to find the approximate value of $b \approx 1.732$ cm.

On your own, try this again using a hypotenuse of 6 feet. Recall that since the $30^\circ - 60^\circ - 90^\circ$ triangle comes from an equilateral triangle, you know that the length of the shorter leg is half the length of the hypotenuse.

Now you should be able to identify the constant ratios in $30^\circ - 60^\circ - 90^\circ$ triangles. The hypotenuse will always be **twice** the length of the shorter leg, and the longer leg is always the product of the length of the shorter leg and $\sqrt{3}$. In ratio form, the sides, in order from shortest to longest are in the ratio $x : x\sqrt{3} : 2x$.

Example 4

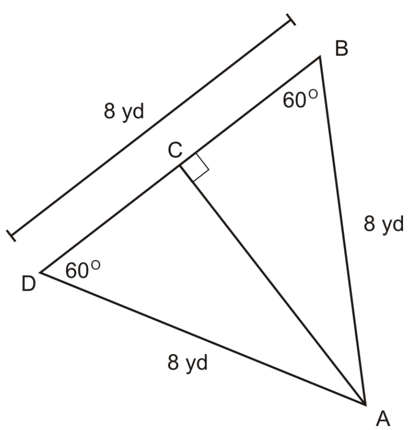
What is the length of the missing leg in the triangle below?



Since the length of the longer leg is the product of the shorter leg and $\sqrt{3}$, you can easily calculate this length. The short leg is 8 inches, so the longer leg will be $8\sqrt{3}$ inches, or about 13.86 inches.

Example 5

What is AC below?



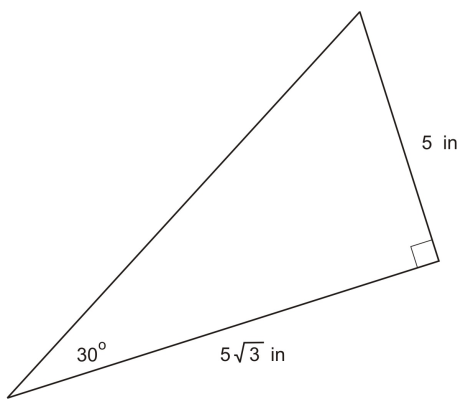
To find the length of segment $\overline{AC}$, identify its relationship to the rest of the triangle. Since it is an altitude, it forms two congruent triangles with angles measuring 30°, 60°, and 90°. So, AC will be the product of BC (the shorter leg) and $\sqrt{3}$.

$$\begin{aligned} AC &= BC\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

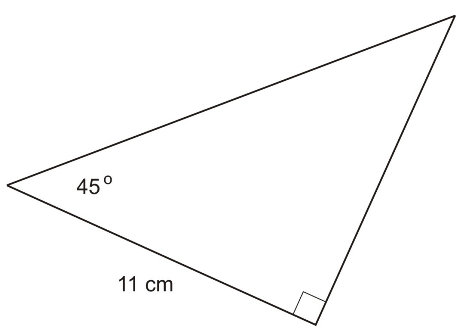
$AC = 4\sqrt{3}$ yards , or approximately 6.93 yards.

Review Questions

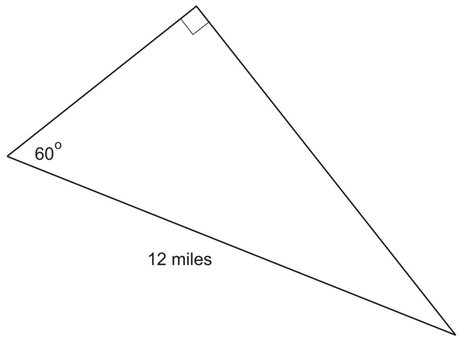
1. Mildred had a piece of scrap wood cut into an equilateral triangle. She wants to cut it into two smaller congruent triangles. What will be the angle measurement of the triangles that result?
2. Roberto has a square pizza. He wants to cut two congruent triangles out of the pizza without leaving any leftovers. What will be the angle measurements of the triangles that result?
3. What is the length of the hypotenuse in the triangle below?



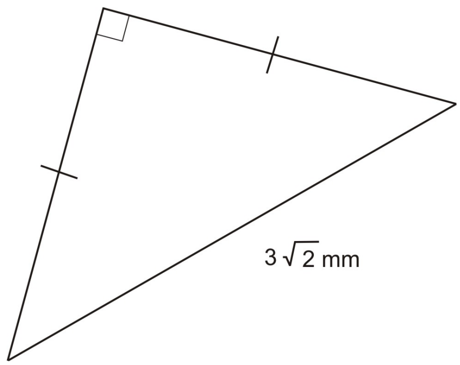
4. What is the length of the hypotenuse in the triangle below?



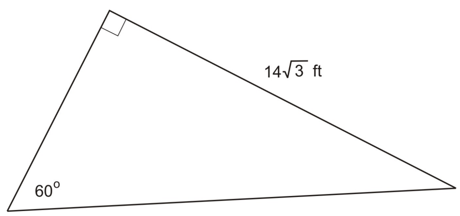
5. What is the length of the longer leg in the triangle below?



6. What is the length of one of the legs in the triangle below?



7. What is the length of the shorter leg in the triangle below?



8. A square window has a diagonal of $5\sqrt{2}$ feet. What is the length of one of its sides?
9. A square block of foam is cut into two congruent wedges. If a side of the original block was 3 feet, how long is the diagonal cut?
10. They wants to find the area of an equilateral triangle but only knows that the length of one side is 6 inches. What is the height of Thuy's triangle? What is the area of the triangle?

Review Answers

1. $30°, 60°$, and $90°$
2. $45°, 45°$, and $90°$
3. 10
4. $11\sqrt{2}$ cm or approx. 15.56 cm
5. $6\sqrt{3}$ miles or approx. 10.39 miles
6. 3 mm
7. 14 feet
8. 5 feet
9. $3\sqrt{2}$ feet or approx. 4.24 feet
10. $3\sqrt{3}$ inches or approx. 5.2 in.

    The area is $9\sqrt{3} \approx 15.59$ inches^2