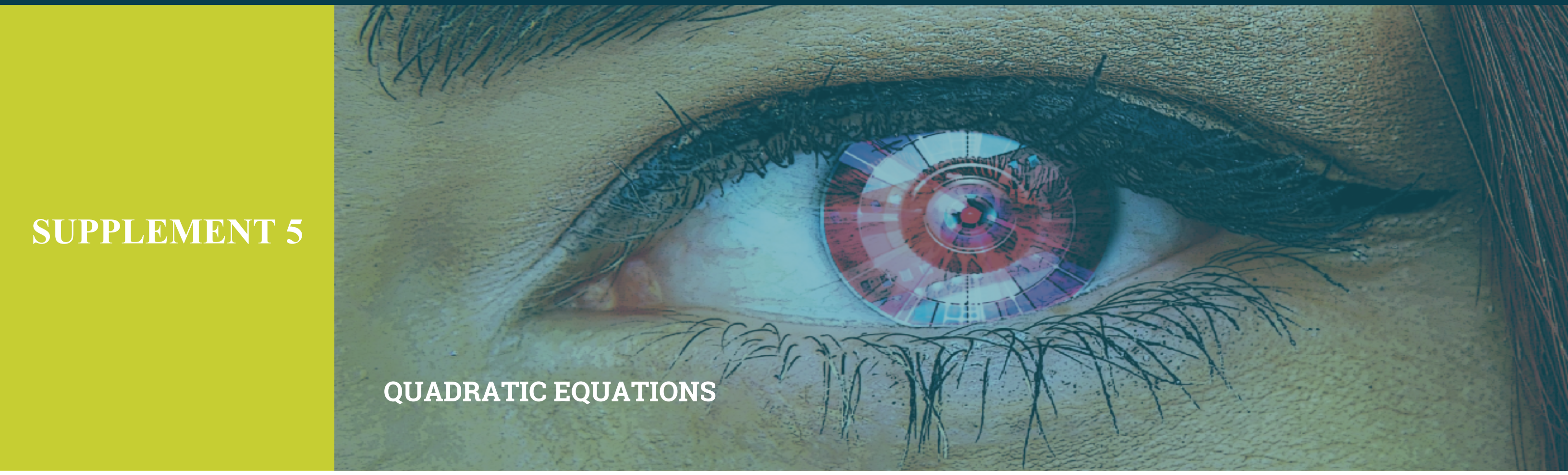


Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: “intographics”/Pixabay)

## Chapter Outline

Solve Quadratic Equations Using the Square Root Property
Solve Quadratic Equations Using the Quadratic Formula

## Introduction

Blink your eyes. You’ve taken a photo. That’s what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

## 9.1 Solve Quadratic Equations Using the Square Root Property

### Learning Objectives

**By the end of this section, you will be able to:**

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

#### Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
2. Simplify: $\sqrt{\frac{32}{5}}$.

3. Factor: $9x^2 - 12x + 4$.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form $ax^2$. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

## Solve Quadratic Equations of the form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

| | |
|---|---|
| | $x^2 = 9$ |
| Put the equation in standard form. | $x^2 - 9 = 0$ |
| Factor the diffe ence of squares. | $(x-3)(x+3) = 0$ |
| Use the Zero Product Property. | $x - 3 = 0 \quad x - 3 = 0$ |
| Solve each equation. | $x = 3 \quad x = -3$ |

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4,\ x = -4$ and $x = 5,\ x = -5$.

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of $k$. This leads to the **Square Root Property**.

**Square Root Property**

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of $k$ and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as *x* equals positive or negative the square root of *k*.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

| | |
|---|---|
| | $x^2 = 9$ |
| Use the Square Root Property. | $x = \pm\sqrt{9}$ |
| | $x = \pm 3$ |
| | So $x = 3$ or $x = -3$. |

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

| | |
|---|---|
| | $x^2 = 7$ |
| Use the Square Root Property. | $x = \sqrt{7}, \quad x = -\sqrt{7}$ |

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

**EXAMPLE 9.1** HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $AX^2 = K$ USING THE SQUARE ROOT PROPERTY

Solve: $x^2 - 50 = 0$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Isolate the quadratic term and make its coefficient one. | Add 50 to both sides to get $x^2$ by itself. | $x^2 - 50 = 0$<br>$x^2 = 50$ |
| **Step 2.** Use Square Root Property. | Remember to write the ± symbol. | $x = \pm\sqrt{50}$ |
| **Step 3.** Simplify the radical. | Rewrite to show two solutions. | $x = \pm\sqrt{25} \cdot \sqrt{2}$<br>$x = \pm 5\sqrt{2}$<br>$x = 5\sqrt{2}, x = -5\sqrt{2}$ |
| **Step 4.** Check the solutions. | Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$ | $x^2 - 50 = 0$<br>$\left(5\sqrt{2}\right)^2 - 50 \stackrel{?}{=} 0$<br>$25 \cdot 2 - 50 \stackrel{?}{=} 0$<br>$0 = 0\ ✓$<br>$x^2 - 50 = 0$<br>$\left(-5\sqrt{2}\right)^2 - 50 \stackrel{?}{=} 0$<br>$25 \cdot 2 - 50 \stackrel{?}{=} 0$<br>$0 = 0\ ✓$ |

**TRY IT : :** 9.1 Solve: $x^2 - 48 = 0.$

**TRY IT : :** 9.2 Solve: $y^2 - 27 = 0.$

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

**HOW TO : :** SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

Step 1. Isolate the quadratic term and make its coefficient one.
Step 2. Use Square Root Property.
Step 3. Simplify the radical.
Step 4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

**EXAMPLE 9.2**

Solve: $3z^2 = 108.$

## Solution

| | |
|---|---|
| | $3z^2 = 108$ |
| The quadratic term is isolated. Divide by 3 to make its coefficient 1. | $\frac{3z^2}{3} = \frac{108}{3}$ |
| Simplify. | $z^2 = 36$ |
| Use the Square Root Property. | $z = \pm\sqrt{36}$ |
| Simplify the radical. | $z = \pm 6$ |
| Rewrite to show two solutions. | $z = 6, \quad z = -6$ |

Check the solutions:

| | |
|---|---|
| $3z^2 = 108$ | $3z^2 = 108$ |
| $3(6)^2 \stackrel{?}{=} 108$ | $3(-6)^2 \stackrel{?}{=} 108$ |
| $3(36) \stackrel{?}{=} 108$ | $3(36) \stackrel{?}{=} 108$ |
| $108 = 108$ ✓ | $108 = 108$ ✓ |

**TRY IT : :** 9.3 Solve: $2x^2 = 98$.

**TRY IT : :** 9.4 Solve: $5m^2 = 80$.

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

**EXAMPLE 9.3**

Solve: $x^2 + 72 = 0$.

## Solution

| | |
|---|---|
| | $x^2 + 72 = 0$ |
| Isolate the quadratic term. | $x^2 = -72$ |
| Use the Square Root Property. | $x = \pm\sqrt{-72}$ |
| Simplify using complex numbers. | $x = \pm\sqrt{72}\, i$ |
| Simplify the radical. | $x = \pm 6\sqrt{2}\, i$ |
| Rewrite to show two solutions. | $x = 6\sqrt{2}\, i, \quad x = -6\sqrt{2}\, i$ |

Check the solutions:

$$\begin{aligned} x^2 + 72 &= 0 \\ (6\sqrt{2}\,i)^2 + 72 &\stackrel{?}{=} 0 \\ 6^2(\sqrt{2})^2 i^2 + 72 &\stackrel{?}{=} 0 \\ 36 \cdot 2 \cdot (-1) + 72 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned} \qquad \begin{aligned} x^2 + 72 &= 0 \\ (6\sqrt{2}\,i)^2 + 72 &\stackrel{?}{=} 0 \\ (-6)^2(\sqrt{2})^2 i^2 + 72 &\stackrel{?}{=} 0 \\ 36 \cdot 2 \cdot (-1) + 72 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

**TRY IT : :** 9.5 Solve: $c^2 + 12 = 0$.

**TRY IT : :** 9.6 Solve: $q^2 + 24 = 0$.

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

## EXAMPLE 9.4

Solve: $\frac{2}{3}u^2 + 5 = 17$.

### Solution

| | |
|---|---|
| | $\frac{2}{3}u^2 + 5 = 17$ |
| Isolate the quadratic term. | $\frac{2}{3}u^2 = 12$ |
| Multiply by $\frac{3}{2}$ to make the coefficient 1. | $\frac{3}{2} \cdot \frac{2}{3}u^2 = \frac{3}{2} \cdot 12$ |
| Simplify. | $u^2 = 18$ |
| Use the Square Root Property. | $u = \pm\sqrt{18}$ |
| Simplify the radical. | $u = \pm\sqrt{9 \cdot 2}$ |
| Simplify. | $u = \pm 3\sqrt{2}$ |
| Rewrite to show two solutions. | $u = 3\sqrt{2}, \quad u = -3\sqrt{2}$ |

Check:

$$\begin{aligned} \tfrac{2}{3}u^2 + 5 &= 17 \\ \tfrac{2}{3}(3\sqrt{2})^2 + 5 &\stackrel{?}{=} 17 \\ \tfrac{2}{3} \cdot 18 + 5 &\stackrel{?}{=} 17 \\ 12 + 5 &\stackrel{?}{=} 17 \\ 17 &= 17 \checkmark \end{aligned} \qquad \begin{aligned} \tfrac{2}{3}u^2 + 5 &= 17 \\ \tfrac{2}{3}(-3\sqrt{2})^2 + 5 &\stackrel{?}{=} 17 \\ \tfrac{2}{3} \cdot 18 + 5 &\stackrel{?}{=} 17 \\ 12 + 5 &\stackrel{?}{=} 17 \\ 17 &= 17 \checkmark \end{aligned}$$

**TRY IT : :** 9.7 Solve: $\frac{1}{2}x^2 + 4 = 24$.

**TRY IT : :** 9.8 Solve: $\frac{3}{4}y^2 - 3 = 18.$

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

**EXAMPLE 9.5**

Solve: $2x^2 - 8 = 41.$

### Solution

| | $2x^2 - 8 = 41$ |
|---|---|
| Isolate the quadratic term. | $2x^2 = 49$ |
| Divide by 2 to make the coefficient 1. | $\frac{2x^2}{2} = \frac{49}{2}$ |
| Simplify. | $x^2 = \frac{49}{2}$ |
| Use the Square Root Property. | $x = \pm\sqrt{\frac{49}{2}}$ |
| Rewrite the radical as a fraction of square roots. | $x = \pm\frac{\sqrt{49}}{\sqrt{2}}$ |
| Rationalize the denominator. | $x = \pm\frac{\sqrt{49} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$ |
| Simplify. | $x = \pm\frac{7\sqrt{2}}{2}$ |
| Rewrite to show two solutions. | $x = \frac{7\sqrt{2}}{2}, \quad x = -\frac{7\sqrt{2}}{2}$ |
| Check:<br>We leave the check for you. | |

**TRY IT : :** 9.9 Solve: $5r^2 - 2 = 34.$

**TRY IT : :** 9.10 Solve: $3t^2 + 6 = 70.$

## Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x - h)^2 = k$ as well. Notice that the quadratic term, $x$, in the original form $ax^2 = k$ is replaced with $(x - h)$.

$$ax^2 = k \qquad a(x-h)^2 = k$$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of $a$, then the Square Root Property can be used on $(x - h)^2$.

**EXAMPLE 9.6**

Solve: $4(y - 7)^2 = 48.$

## Solution

| | |
|---|---|
| | $4(y-7)^2 = 48$ |
| Divide both sides by the coefficient 4. | $(y-7)^2 = 12$ |
| Use the Square Root Property on the binomial | $y - 7 = \pm\sqrt{12}$ |
| Simplify the radical. | $y - 7 = \pm 2\sqrt{3}$ |
| Solve for $y$. | $y = 7 \pm 2\sqrt{3}$ |
| Rewrite to show two solutions. | $y = 7 + 2\sqrt{3}, \quad y = 7 - 2\sqrt{3}$ |

Check:

$$\begin{aligned} 4(y-7)^2 &= 48 \\ 4\left(7+2\sqrt{3}-7\right)^2 &\stackrel{?}{=} 48 \\ 4\left(2\sqrt{3}\right)^2 &\stackrel{?}{=} 48 \\ 4(12) &\stackrel{?}{=} 48 \\ 48 &= 48 \checkmark \end{aligned} \qquad \begin{aligned} 4(y-7)^2 &= 48 \\ 4\left(7-2\sqrt{3}-7\right)^2 &\stackrel{?}{=} 48 \\ 4\left(-2\sqrt{3}\right)^2 &\stackrel{?}{=} 48 \\ 4(12) &\stackrel{?}{=} 48 \\ 48 &= 48 \checkmark \end{aligned}$$

**TRY IT : :** 9.11 Solve: $3(a-3)^2 = 54$.

**TRY IT : :** 9.12 Solve: $2(b+2)^2 = 80$.

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

**EXAMPLE 9.7**

Solve: $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$.

## Solution

| | |
|---|---|
| | $\left(x-\frac{1}{3}\right)^2 = \frac{5}{9}$ |
| Use the Square Root Property. | $x-\frac{1}{3} = \pm\sqrt{\frac{5}{9}}$ |
| Rewrite the radical as a fraction of square roots. | $x-\frac{1}{3} = \pm\frac{\sqrt{5}}{\sqrt{9}}$ |
| Simplify the radical. | $x-\frac{1}{3} = \pm\frac{\sqrt{5}}{3}$ |
| Solve for $x$. | $x = \frac{1}{3}\pm\frac{\sqrt{5}}{3}$ |
| Rewrite to show two solutions. | $x=\frac{1}{3}+\frac{\sqrt{5}}{3},\quad x=\frac{1}{3}-\frac{\sqrt{5}}{3}$ |

Check:
We leave the check for you.

**TRY IT : :** 9.13 Solve: $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

**TRY IT : :** 9.14 Solve: $\left(y+\frac{3}{4}\right)^2=\frac{7}{16}$.

We will start the solution to the next example by isolating the binomial term.

**EXAMPLE 9.8**

Solve: $2(x-2)^2+3=57$.

## Solution

| | |
|---|---|
| | $2(x-2)^2+3 = 57$ |
| Subtract 3 from both sides to isolate the binomial term. | $2(x-2)^2 = 54$ |
| Divide both sides by 2. | $(x-2)^2 = 27$ |
| Use the Square Root Property. | $x-2 = \pm\sqrt{27}$ |
| Simplify the radical. | $x-2 = \pm 3\sqrt{3}$ |
| Solve for $x$. | $x = 2\pm 3\sqrt{3}$ |
| Rewrite to show two solutions. | $x=2+3\sqrt{3},\quad x=2-3\sqrt{3}$ |

Check:
We leave the check for you.

**TRY IT : :** 9.15 Solve: $5(a-5)^2+4=104$.

**TRY IT : :** 9.16 Solve: $3(b+3)^2-8=88$.

Sometimes the solutions are complex numbers.

**EXAMPLE 9.9**

Solve: $(2x-3)^2 = -12.$

### Solution

| | |
|---|---|
| | $(2x-3)^2 = -12$ |
| Use the Square Root Property. | $2x-3 = \pm\sqrt{-12}$ |
| Simplify the radical. | $2x-3 = \pm 2\sqrt{3}\,i$ |
| Add 3 to both sides. | $2x = 3 \pm 2\sqrt{3}\,i$ |
| Divide both sides by 2. | $x = \dfrac{3 \pm 2\sqrt{3}\,i}{2}$ |
| Rewrite in standard form. | $x = \dfrac{3}{2} \pm \dfrac{2\sqrt{3}\,i}{2}$ |
| Simplify. | $x = \dfrac{3}{2} \pm \sqrt{3}\,i$ |
| Rewrite to show two solutions. | $x = \dfrac{3}{2} + \sqrt{3}\,i, \quad x = \dfrac{3}{2} - \sqrt{3}\,i$ |

Check:
We leave the check for you.

**TRY IT : :** 9.17 Solve: $(3r+4)^2 = -8.$

**TRY IT : :** 9.18 Solve: $(2t-8)^2 = -10.$

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

**EXAMPLE 9.10**

Solve: $4n^2 + 4n + 1 = 16.$

### Solution

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

| | |
|---|---|
| | $4n^2 + 4n + 1 = 16$ |
| Factor the perfect square trinomial. | $(2n+1)^2 = 16$ |
| Use the Square Root Property. | $2n+1 = \pm\sqrt{16}$ |
| Simplify the radical. | $2n+1 = \pm 4$ |
| Solve for $n$. | $2n = -1 \pm 4$ |
| Divide each side by 2. | $\dfrac{2n}{2} = \dfrac{-1 \pm 4}{2}$ <br> $n = \dfrac{-1 \pm 4}{2}$ |
| Rewrite to show two solutions. | $n = \dfrac{-1+4}{2}, \quad n = \dfrac{-1-4}{2}$ |
| Simplify each equation. | $n = \dfrac{3}{2}, \qquad n = -\dfrac{5}{2}$ |

Check:

$$4n^2 + 4n + 1 = 16 \qquad\qquad 4n^2 + 4n + 1 = 16$$

$$4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad\qquad 4\left(-\frac{5}{2}\right)^2 + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$

$$4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad\qquad 4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$

$$9 + 6 + 1 \stackrel{?}{=} 16 \qquad\qquad 25 - 10 + 1 \stackrel{?}{=} 16$$

$$16 = 16 ✓ \qquad\qquad 16 = 16 ✓$$

**TRY IT : :** 9.19 Solve: $9m^2 - 12m + 4 = 25.$

**TRY IT : :** 9.20 Solve: $16n^2 + 40n + 25 = 4.$

**MEDIA : :**

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- **Solving Quadratic Equations: The Square Root Property (https://openstax.org/l/37SqRtProp1)**
- **Using the Square Root Property to Solve Quadratic Equations (https://openstax.org/l/37SqRtProp2)**

## 9.1 EXERCISES

### Practice Makes Perfect

#### Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

*In the following exercises, solve each equation.*

**1.** $a^2 = 49$

**3.** $r^2 - 24 = 0$

**4.** $t^2 - 75 = 0$

**6.** $v^2 - 80 = 0$

**7.** $4m^2 = 36$

**9.** $\frac{4}{3}x^2 = 48$

**10.** $\frac{5}{3}y^2 = 60$

**12.** $y^2 + 64 = 0$

**13.** $x^2 + 63 = 0$

**15.** $\frac{4}{3}x^2 + 2 = 110$

**16.** $\frac{2}{3}y^2 - 8 = -2$

**18.** $\frac{3}{2}b^2 - 7 = 41$

**19.** $7p^2 + 10 = 26$

**21.** $5y^2 - 7 = 25$

#### Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

*In the following exercises, solve each equation.*

**23.** $(u - 6)^2 = 64$

**25.** $(m - 6)^2 = 20$

**26.** $(n + 5)^2 = 32$

**28.** $\left(x + \frac{1}{5}\right)^2 = \frac{7}{25}$

**29.** $\left(y + \frac{2}{3}\right)^2 = \frac{8}{81}$

**31.** $(a - 7)^2 + 5 = 55$

**32.** $(b - 1)^2 - 9 = 39$

**34.** $5(x + 3)^2 - 7 = 68$

**35.** $(5c + 1)^2 = -27$

**37.** $(4x - 3)^2 + 11 = -17$

**38.** $(2y + 1)^2 - 5 = -23$

**40.** $n^2 + 8n + 16 = 27$

**41.** $x^2 - 6x + 9 = 12$

**43.** $25x^2 - 30x + 9 = 36$

**44.** $9y^2 + 12y + 4 = 9$

**46.** $64x^2 + 144x + 81 = 25$

# 9.3 Solve Quadratic Equations Using the Quadratic Formula

## Learning Objectives

**By the end of this section, you will be able to:**

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

### Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $b^2 - 4ab$ when $a = 3$ and $b = -2$.
2. Simplify: $\sqrt{108}$.
3. Simplify: $\sqrt{50}$.

## Solve Quadratic Equations Using the Quadratic Formula

Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for *x*.

We start with the standard form of a quadratic equation and solve it for *x* by completing the square.

| | |
|---|---|
| | $ax^2 + bx + c = 0 \quad a \neq 0$ |
| Isolate the variable terms on one side. | $ax^2 + bx = -c$ |
| Make the coefficient of $x^2$ equal to 1, by dividing by *a*. | $\frac{ax^2}{a} + \frac{b}{a}x = -\frac{c}{a}$ |
| Simplify. | $x^2 + \frac{b}{a}x = -\frac{c}{a}$ |
| To complete the square, find $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ and add it to both sides of the equation. | |
| $\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$ | $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + = -\frac{c}{a} + \frac{b^2}{4a^2}$ |
| The left side is a perfect square, factor it. | $\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$ |
| Find the common denominator of the right side and write equivalent fractions with the common denominator. | $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \cdot 4a}{a \cdot 4a}$ |

| | |
|---|---|
| Simplify. | $\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$ |
| Combine to one fraction. | $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$ |
| Use the square root property. | $x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$ |
| Simplify the radical. | $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$ |
| Add $-\frac{b}{2a}$ to both sides of the equation. | $x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$ |
| Combine the terms on the right side. | $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ |
| | This equation is the Quadratic Formula. |

### Quadratic Formula

The solutions to a quadratic equation of the form $ax^2+bx+c=0$, where $a \neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of *a*, *b*, and *c* from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the formula is an equation. Make sure you use both sides of the equation.

**EXAMPLE 9.21** HOW TO SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

## Solution

| | | |
|---|---|---|
| **Step 1.** Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values. | This equation is in standard form. | $ax^2+bx+c=0$<br>$2x^2+9x-5=0$<br>$a=2, b=9, c=-5$ |
| **Step 2.** Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*. | Substitute in $a=2$, $b=9$, $c=-5$ | $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$<br>$x=\frac{-9\pm\sqrt{9^2-4\cdot2\cdot(-5)}}{2\cdot2}$ |

| | | |
|---|---|---|
| **Step 3.** Simplify the fraction, and solve for *x*. | | $x = \frac{-9 \pm \sqrt{81-(-40)}}{4}$<br>$x = \frac{-9 \pm \sqrt{121}}{4}$<br>$x = \frac{-9 \pm 11}{4}$<br>$x = \frac{-9+11}{4}$ $\quad x = \frac{-9-11}{4}$<br>$x = \frac{2}{4}$ $\quad x = \frac{-20}{4}$<br>$x = \frac{1}{2}$ $\quad x = -5$ |
| **Step 4.** Check the solutions. | Put each answer in the original equation to check.<br>Substitute $x = \frac{1}{2}$.<br><br>Substitute $x = -5$. | $2x^2 + 9x - 5 = 0$<br>$2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$<br>$2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$<br>$2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$<br>$\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$<br>$\frac{10}{2} - 5 \stackrel{?}{=} 0$<br>$5 - 5 \stackrel{?}{=} 0$<br>$0 = 0\,✓$<br><br>$2x^2 + 9x - 5 = 0$<br>$2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$<br>$2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$<br>$50 - 45 - 5 \stackrel{?}{=} 0$<br>$0 = 0\,✓$ |

**TRY IT : :** 9.41 Solve by using the Quadratic Formula: $3y^2 - 5y + 2 = 0$.

**TRY IT : :** 9.42 Solve by using the Quadratic Formula: $4z^2 + 2z - 6 = 0$.

**HOW TO : :** SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA.

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of *a*, *b*, and *c*.

Step 2. Write the Quadratic Formula. Then substitute in the values of *a*, *b*, and *c*.

Step 3. Simplify.

Step 4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic

Formula is an EQUATION. Be sure you start with "$x =$".

**EXAMPLE 9.22**

Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

## Solution

| | |
|---|---|
| | $x^2 - 6x = -5$ |
| Write the equation in standard form by adding 5 to each side. | $x^2 - 6x + 5 = 0$ |
| This equation is now in standard form. | $ax^2 + bx + c = 0$<br>$x^2 - 6x + 5 = 0$ |
| Identify the values of $a,\ b,\ c$. | $a = 1,\ b = -6,\ c = 5$ |
| Write the Quadratic Formula. | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a,\ b,\ c$. | $x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$ |
| Simplify. | $x = \frac{6 \pm \sqrt{36 - 20}}{2}$<br>$x = \frac{6 \pm \sqrt{16}}{2}$<br>$x = \frac{6 \pm 4}{2}$ |
| Rewrite to show two solutions. | $x = \frac{6 + 4}{2},\quad x = \frac{6 - 4}{2}$ |
| Simplify. | $x = \frac{10}{2},\quad x = \frac{2}{2}$ |
| | $x = 5,\quad x = 1$ |

Check:

$$\begin{array}{rr} x^2 - 6x + 5 = 0 & x^2 - 6x + 5 = 0 \\ 5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0 & 1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0 \\ 25 - 30 + 5 \stackrel{?}{=} 0 & 1 - 6 + 5 \stackrel{?}{=} 0 \\ 0 = 0 ✓ & 0 = 0 ✓ \end{array}$$

**TRY IT : :** 9.43 Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

**TRY IT : :** 9.44 Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

**EXAMPLE 9.23**

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

## Solution

| | $2x^2 + 10x + 11 = 0$ |
|---|---|
| This equation is in standard form. | $\begin{matrix} ax^2 & + & bx & + & c & = 0 \\ 2x^2 & + & 10x & + & 11 & = 0 \end{matrix}$ |
| Identify the values of *a*, *b*, and *c*. | $a = 2,\ b = 10,\ c = 11$ |
| Write the Quadratic Formula. | $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of *a*, *b*, and *c*. | $x = \dfrac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$ |
| Simplify. | $x = \dfrac{-10 \pm \sqrt{100 - 88}}{4}$ |
| | $x = \dfrac{-10 \pm \sqrt{12}}{4}$ |
| Simplify the radical. | $x = \dfrac{-10 \pm 2\sqrt{3}}{4}$ |
| Factor out the common factor in the numerator. | $x = \dfrac{2(-5 \pm \sqrt{3})}{4}$ |
| Remove the common factors. | $x = \dfrac{-5 \pm \sqrt{3}}{2}$ |
| Rewrite to show two solutions. | $x = \dfrac{-5 + \sqrt{3}}{2},\quad x = \dfrac{-5 - \sqrt{3}}{2}$ |
| Check:<br>We leave the check for you! | |

**TRY IT : :** 9.45 Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

**TRY IT : :** 9.46 Solve by using the Quadratic Formula: $5n^2 + 4n - 4 = 0$.

When we substitute *a*, *b*, and *c* into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

**EXAMPLE 9.24**

Solve by using the Quadratic Formula: $3p^2 + 2p + 9 = 0$.

## Solution

| | $3p^2 + 2p + 9 = 0$ |
|---|---|
| This equation is in standard form | $\begin{matrix} ax^2 & + & bx & + & c & = 0 \\ 3p^2 & + & 2p & + & 9 & = 0 \end{matrix}$ |
| Identify the values of $a, b, c$. | $a = 3,\ b = 2,\ c = 9$ |
| Write the Quadratic Formula. | $p = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of $a, b, c$. | $p = \dfrac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$ |

| | |
|---|---|
| Simplify. | $p=\frac{-2\pm\sqrt{4-108}}{6}$ |
| | $p=\frac{-2\pm\sqrt{-104}}{6}$ |
| Simplify the radical using complex numbers. | $p=\frac{-2\pm\sqrt{104}\,i}{6}$ |
| Simplify the radical. | $p=\frac{-2\pm2\sqrt{26}\,i}{6}$ |
| Factor the common factor in the numerator. | $p=\frac{2(-1\pm\sqrt{26}\,i)}{6}$ |
| Remove the common factors. | $p=\frac{-1\pm\sqrt{26}\,i}{3}$ |
| Rewrite in standard $a+bi$ form. | $p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$ |
| Write as two solutions. | $p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$ |

**TRY IT : :** 9.47 Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

**TRY IT : :** 9.48 Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

**EXAMPLE 9.25**

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

## Solution

Our first step is to get the equation in standard form.

| | |
|---|---|
| | $x(x+6)+4=0$ |
| Distribute to get the equation in standard form. | $x^2+6x+4=0$ |
| This equation is now in standard form | $\begin{array}{l} ax^2+bx+c=0 \\ x^2+6x+4=0 \end{array}$ |
| Identify the values of $a, b, c$. | $a=1,\ b=6,\ c=4$ |
| Write the Quadratic Formula. | $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ |
| Then substitute in the values of $a, b, c$. | $x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot1\cdot(4)}}{2\cdot1}$ |
| Simplify. | $x=\frac{-6\pm\sqrt{36-16}}{2}$ |
| | $x=\frac{-6\pm\sqrt{20}}{2}$ |
| Simplify the radical. | $x=\frac{-6\pm2\sqrt{5}}{2}$ |

| | |
|---|---|
| Factor the common factor in the numerator. | $x = \frac{2\left(-3 \pm 2\sqrt{5}\right)}{2}$ |
| Remove the common factors. | $x = -3 \pm 2\sqrt{5}$ |
| Write as two solutions. | $x = -3 + 2\sqrt{5},\ \ x = -3 - 2\sqrt{5}$ |
| Check:<br>We leave the check for you! | |

**TRY IT : :** 9.49 Solve by using the Quadratic Formula: $x(x+2) - 5 = 0$.

**TRY IT : :** 9.50 Solve by using the Quadratic Formula: $3y(y-2) - 3 = 0$.

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

**EXAMPLE 9.26**

Solve by using the Quadratic Formula: $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$.

### Solution

Our first step is to clear the fractions.

| | |
|---|---|
| | $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$ |
| Multiply both sides by the LCD, 6, to clear the fractions. | $6\left(\frac{1}{2}u^2 + \frac{2}{3}u\right) = 6\left(\frac{1}{3}\right)$ |
| Multiply. | $3u^2 + 4u = 2$ |
| Subtract 2 to get the equation in standard form. | $\begin{array}{l} ax^2 + bx + c = 0 \\ 3u^2 + 4u - 2 = 0 \end{array}$ |
| Identify the values of *a*, *b*, and *c*. | $a = 3,\ b = 4,\ c = -2$ |
| Write the Quadratic Formula. | $u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| Then substitute in the values of *a*, *b*, and *c*. | $u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$ |
| Simplify. | $u = \frac{-4 \pm \sqrt{16 + 24}}{6}$ |
| | $u = \frac{-4 \pm \sqrt{40}}{6}$ |
| Simplify the radical. | $u = \frac{-4 \pm 2\sqrt{10}}{6}$ |
| Factor the common factor in the numerator. | $u = \frac{2\left(-2 \pm \sqrt{10}\right)}{6}$ |
| Remove the common factors. | $u = \frac{-2 \pm \sqrt{10}}{3}$ |
| Rewrite to show two solutions. | $u = \frac{-2 + \sqrt{10}}{3},\qquad u = \frac{-2 - \sqrt{10}}{3}$ |

## 9.3 EXERCISES

### Practice Makes Perfect

#### Solve Quadratic Equations Using the Quadratic Formula

*In the following exercises, solve by using the Quadratic Formula.*

**113.** $4m^2 + m - 3 = 0$

**115.** $2p^2 - 7p + 3 = 0$

**116.** $3q^2 + 8q - 3 = 0$

**118.** $q^2 + 3q - 18 = 0$

**119.** $r^2 - 8r = 33$

**121.** $3u^2 + 7u - 2 = 0$

**122.** $2p^2 + 8p + 5 = 0$

**124.** $5b^2 + 2b - 4 = 0$

**125.** $x^2 + 8x - 4 = 0$

**127.** $3y^2 + 5y - 2 = 0$

**128.** $6x^2 + 2x - 20 = 0$

**130.** $2x^2 - x + 1 = 0$

**131.** $8x^2 - 6x + 2 = 0$

**133.** $(v + 1)(v - 5) - 4 = 0$

**134.** $(x + 1)(x - 3) = 2$

**136.** $(x + 2)(x + 6) = 21$

**137.** $\frac{1}{3}m^2 + \frac{1}{12}m = \frac{1}{4}$

**139.** $\frac{3}{4}b^2 + \frac{1}{2}b = \frac{3}{8}$

#### Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

*In the following exercises, determine the number of solutions for each quadratic equation.*

**145.**

ⓐ $4x^2 - 5x + 16 = 0$

ⓑ $36y^2 + 36y + 9 = 0$

ⓒ $6m^2 + 3m - 5 = 0$

**146.**

ⓐ $9v^2 - 15v + 25 = 0$

ⓑ $100w^2 + 60w + 9 = 0$

ⓒ $5c^2 + 7c - 10 = 0$

**147.**

ⓐ $r^2 + 12r + 36 = 0$

ⓑ $8t^2 - 11t + 5 = 0$

ⓒ $3v^2 - 5v - 1 = 0$